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GENERIC TRIANGLES IN HYPERBOLIC AND SPHERICAL GEOMETRY

THESIS IN GEOMETRY

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Senza la geometria la vita
non ha punti di riferimento.

Contents

Introduction

In this thesis we will be working with groups of triangles focusing more in non Euclidean geometries, precisely, the spherical and the hyperbolic geometry. A *triangle group* is a group G_τ generated by the reflections r_1, r_2, r_3 across the sides of an Euclidean, spherical or hyperbolic triangle τ with internal angles $\alpha_i = \pi/n_i$, where $1/n_1 + 1/n_2 + 1/n_3$ being $= 1, > 1$ or < 1 determines the geometry and it's properties. Based on the fact that the triangle τ tiles the Euclidean plane, the sphere or the hyperbolic plane, we have the finite presentation $G_\tau = \langle x_1, x_2, x_3 \mid x_1^2, x_2^2, x_3^2, (x_2x_3)^{n_1}, (x_3x_1)^{n_2}, (x_1x_2)^{n_3} \rangle$ where x_i corresponds to the reflection r_i.

We will start our study with some generalities about Riemannian Manifolds and metrics in coordinates in Chapter 1. We will then define some important definitions and state some of the most important theorems regarding connections, geodesics, curvature and isometries of Riemann manifolds. After that we will define the homogeneous and isotropic manifolds and give some important hints of conformal geometry focusing on circular inversions, which will later be used to calculate the curvature of the considerated manifolds.
In the second section of Chapter 1 we will consider the models of the Euclidean Space $\mathbb{R}^n$, the Sphere S_r^n and the Hyperbolic Space H_r^n. We will use the technique of stereographic projection in order to give the metric of the spherical space. We will then see how the orthogonal group $O(n+1)$ on $\mathbb{R}^{n+1}$ preserves S_r^n and the Euclidean metric, so its restriction to S_r^n acts by isometries of the sphere, and the subgroup $O_+(n+1)$ of $O(n+1)$ preserves $\mathbb{H}_r^n$, and because it preserves m it acts on $\mathbb{H}_r^n$ as isometries. We will then see how are the geodesics of each model space represented. Finally, basing on the fact that both S_r^n and $\mathbb{H}_r^n$ are homogeneous and isotropic we will show that they have constant curvatures, $\frac{1}{r^2}$ and $-\frac{1}{r^2}$ respectively, and give a result about the classification of constant curvature metrics.

In Chapter 2 we will state the most important results of spherical and hyperbolic trigonometry as the laws of sines and cosines and the Spherical Pythagorean theorem. We will then consider the Euclidean plane $\mathbb{R}^2$, the sphere and the hyperbolic plane and show for each one respectively how the triangles are defined. After that we will consider each model space separately and study their tessellations by the triangle groups generated by the reflections accross their sides. The Euclidean plane is best tessellated by $T_\tau(2,3,6)$, $T_\tau(3,3,3)$ and $T_\tau(2,4,4)$. The sphere is best tessellated by the dihedral groups$T_\tau(2,2,n)$, the tetrahedral group $T_\tau(3,3,2)$, the octahedral group $T_\tau(2,3,4)$ and the icosahedral group $T_\tau(2,3,5)$. Finally, we will consider the hyperbolic plane when tessellated by $T_\tau(2,3,0)$, $T_\tau(0,0,0)$, $T_\tau(2,3,7)$ and $T_\tau(2,3,8)$.

In Chapter 3 we will introduce the definition of generic triangle in Euclidean space and state some important results regarding to it. We also define a stable sequence of indices and according to that we prove that the Euclidean space is stable. We will see how the group of translations is made and give some assumptional ideas of how we can proceed in order to obtain analogous results for the hyperbolic plane and the sphere.

Chapter 1

Models of the Euclidean Space $\mathbb{R}^n$, the Sphere S_r^n and the Hyperbolic space $\mathbb{H}_r^n$

1.1 Generalities about Riemannian manifolds and the metrics in coordinates

1.1.1 Riemannian metric and Riemannian manifold

Definition 1.1.1. A *Riemannian metric* on a smooth manifold M is a smoothly chosen inner product $g_x : T_x(M) \times T_x(M) \to \mathbb{R}$ on each of the tangent spaces $T_x(M)$ of M. In other words, for each $x \in M$, $\langle ., . \rangle = g(., .) = g_x(., .)$ satisfies:
(1) $\langle v, w \rangle_x = \langle w, v \rangle_x$ for all $v, w \in T_xM$;
(2) $\langle v, v \rangle_x \geq 0$ for all $v \in T_xM$;
(3) $\langle v, v \rangle_x = 0$ if and only if $v = 0$.

Definition 1.1.2. A manifold together with a given Riemannian metric is called a *Riemannian manifold*.

Example 1.1.3. $\mathbb{R}^n$ with its Euclidean metric $\bar{g}$, which is the usual inner product on each tangent space $T_x\mathbb{R}^n$ under the natural identifcation $T_x\mathbb{R}^n = \mathbb{R}^n$,is an example of Riemannian manifold. In standard coordinates, this can be written in several ways:

$$\bar{g} = \sum_i dx^i dx^i = \sum_i (dx^i)^2 = \sum_{i,j} \delta_{ij} dx^i dx^j. \tag{1.1}$$

The matrix of $\bar{g}$ in these coordinates is thus $\bar{g}_{ij} = \delta_{ij}$.

Many other examples of Riemannian metrics are these of submanifolds, products, and quotients of Riemannian manifolds.
Suppose $(\bar{M}, \bar{g})$ is a Riemannian manifold, and $\imath : M \hookrightarrow \bar{M}$ is an immersed submanifold of $\bar{M}$. The *induced metric* on M is the 2-tensor $g = \imath^* \bar{g}$, which is just the restriction of $\bar{g}$ to vectors tangent to M. Because the restriction of an inner product is itself an inner product, this obviously defines a Riemannian metric on M. For instance, the standard metric on the sphere $S^n_r \subset \mathbb{R}^{n+1}$ is obtained in this way.
Computations on a submanifold are usually done in terms of a local parametrization: this is an embedding of an open subset $\mathcal{U} \subset \mathbb{R}^n$ into $\bar{M}$, whose image is an open subset of M. For example, if $X : \mathcal{U} \to \mathbb{R}^m$ is a parametrization of a submanifold $M \subset \mathbb{R}^m$ with the induced metric, which, in standard coordinates $(u^1, ..., u^n)$ on $\mathcal{U}$ is just

$$g = \sum_{i=1}^{m} (dX^i)^2 = \sum_{i,j=1}^{m} \Big(\frac{\partial X^i}{\partial u^j} du^j \Big)^2 .$$

We will now introduce some preliminary concepts of Riemannian geometry which we will use especially in order to determine the sectional curvature of the Spherical and Hyperbolic spaces.

Lemma 1.1.4. *Let $F : M \to N$ be an immersion and g a Riemannian metric on N.*
(1) $F^* g$ is a Riemannian metric on M.
(2) If $g = g_{ij} dy^i dy^j$ in a coordinate frame on N, then

$$F^*(g)|F^{-1}(U) = g_{ij} dF^i dF^j .$$

Proof. See page 7 of [27]. □

Definition 1.1.5. If $F : U \to M$ is a parametrization (an inverse chart) of an open subset of $M \subset R^m$, then the *pull-back* of the induced metric on M is

$$F^*(\bar{g}) = F^*(\delta_{ij} dx^i dx^j) = \delta_{ij} dF^i dF^j = \sum_{i=1}^{m} (dF^i)^2 .$$

1.1.2 Connections, geodesics and curvature

Definition 1.1.6. Let $E \subset M$ be a smooth vector bundle over M and $\mathcal{E}(M)$ the $C^\infty(M)$-module of smooth sections. A *connection* on E is a map

$$\begin{aligned} \mathcal{T}(M) \times \mathcal{E}(M) &\xrightarrow{\nabla} \mathcal{E}(M) \\ (X, Y) &\to \nabla_X Y \end{aligned}$$

which is $C^\infty(M)$-linear in X and $\mathbb{R}$-linear in Y and satisfies the product rule

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y$$

for all $f \in C^\infty(M)$.

Definition 1.1.7. A *linear connection* is a connection $\mathcal{T}(M) \times \mathcal{T}(M) \xrightarrow{\nabla} \mathcal{T}(M)$ on the tangent bundle of M.

Definition 1.1.8. Let $g : I \to M$ be a smooth curve on M. For any $t \in I$, the tangent vector

$$\dot{g}(t) = g_*(\frac{d}{dt}(t)) \in T_{g(t)}M, \quad \dot{g}f = \frac{d(f \circ g)}{dt}(t), \quad f \in C^\infty$$

is called the *velocity vector* of g at the point $g(t)$.

Definition 1.1.9. A *vector field* along g is a smooth map $V : I \to TM$ such that the diagram commutes. A vector field V along g is extendible

$$\begin{tikzcd} & TM \arrow[d] \\ I \arrow[ur, "V"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$

Figure 1.1: A vector field.

if $V(t) = \overline{V}(g(t))$ for some vector field $\overline{V}$ on a neighborhood of $g(I)$. The $C^\infty(I)$-module of all vector fields along g is denoted $\mathcal{T}(g)$.

Definition 1.1.10. A smooth curve g is a *geodesic* (with respect to the connection ∇) if the covariant derivative of its velocity field vanishes, $D_t\dot{g} = 0$. A geodesic is a curve with constant velocity.

Proposition 1.1.11. ***(Existence and uniqueness of geodesics)*** *Let p be a point on M and $v \in T_pM$ a tangent vector at p. There exists a unique maximal geodesic $g_v : I \to M$ defined on an open interval containing 0 such that $g_v(0) = p$ and $\dot{g}_v(0) = v$*

Proof. See page 17 of [27]. ☐

Lemma 1.1.12. ***(Rescaling lemma)*** *Let $v \in T_pM$ be a tangent vector at $p \in M$ and let $c \in \mathbb{R}$ be a real number. The geodesic g_{cv} is defined at t if and only if the geodesic g_v is defined at ct and then $g_{cv}(t) = g_v(ct)$.*

Proof. See page 17 of [27]. □

Definition 1.1.13. Let M be a manifold and let us define a subset

$$\mathcal{E} := \{v \in TM : \gamma_v \text{ is defined on an interval containing } [0,1]\}$$

and then define the *exponential map* $exp : \mathcal{E} \to M$ by

$$exp(v) = \gamma_v(1).$$

For each $p \in M$, the *restricted exponential map* exp_p is the restriction of exp to the set $\mathcal{E}_p := \mathcal{E} \cap T_pM$.

Proposition 1.1.14. ***(Properties of the Exponential Map)***
(a) $\mathcal{E}$ is an open subset of TM containing the zero section, and each set $\mathcal{E}_p$ is star-shaped with respect to 0.
(b) For each $v \in TM$, the geodesic γ_v is given by

$$\gamma_v(t) = exp(tv)$$

for all t such that either side is defined.
(c) The exponential map is smooth.

Proof. See page 72 of [31]. □

Definition 1.1.15. Geodesic *normal coordinates* are local coordinates on a manifold with an affine connection afforded by the exponential map $exp_p : T_pM \ni v \to M$ and an isomorphism $E : \mathbb{R}^n \to T_pM$ given by any basis of the tangent space at the fixed basepoint $p \in M$. If the additional structure of a Riemannian metric is imposed, then the basis defined by E is required in addition to be orthonormal, and the resulting coordinate system is then known as a *Riemannian normal coordinate system.* Normal coordinates exist on a normal neighborhood of a point $p \in M$. A *normal neighborhood* $\mathcal{U}$ is a subset of M such that there is a proper neighborhood V of the origin in the tangent space T_pM, and exp_p acts as a diffeomorphism between $\mathcal{U}$ and V.

Theorem 1.1.16. ***(The fundamental theorem of Riemannian geometry)*** *A Riemannian manifold admits precisely one symmetric connection compatible with the metric. This connection is called the Riemannian connection or the Levi - Civita connection of g.*

Proof. See page 68 of [31]. □

Definition 1.1.17. The connection ∇ is symmetric if

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for all vector fields $X, Y \in (M)$.

Definition 1.1.18. The connection ∇ is compatible with the metric g if

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle X, \nabla_X Z\rangle$$

for any three vector fields $X, Y, Z \in \mathcal{T}(M)$.

Definition 1.1.19. The *Riemann curvature tensor* is the $(3,1)$-tensor Rm whose action on vector fields is given by $Rm(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle$ corresponding to the $C^\infty(M)$-multilinear map

$$T(M) \times T(M) \times T(M) \xrightarrow{R} T(M),$$

$$(X, Y, Z) \to R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

called the *Riemannian curvature endomorphism.*

In local coordiates (x_i) the components of the curvature tensors

$$R = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l, \quad Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

are given by

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^l \partial_l$$

so that

$$R_{ijkl} = \langle R(\partial_i, \partial_j)\partial_k, \partial_l\rangle = \langle R_{ijk}{}^m \partial_m, \partial_l\rangle = g_{lm} R_{ijk}{}^m, \quad R_{ijk}{}^m = g^{ml} R_{ijkl}.$$

Proposition 1.1.20. *The Riemann curvature tensor Rm is invariant under local isometries.*

Proof. See page 31 of [27]. □

Definition 1.1.21. *Ricci curvature* is the trace on the first and last variable of the Riemann curvature endomorphism: $Rc = tr(R) = tr_g(Rm) \in \mathcal{T}_0^2(M)$. This means that Ricci curvature is the tensor given by

$$Rc(X, Y) = tr(U \to R(U, X)Y).$$

Ricci curvature is a $(2,0)$-tensor with components

$$Rc_{ij} = R_{kij}{}^k = g^{kl} R_{kijl} = g^{lk} R_{kijl} = R^l{}_{ijl} = R^k{}_{ijk}.$$

Let $(\tilde{M}, g)$ be a Riemannian manifold and $M \subset \tilde{M}$ an embedded submanifold equipped with the induced metric, also called g. The 2nd fundamental form of the Riemannian submanifold M is the difference between the Riemannian connections $\tilde{\nabla}$ and ∇. (The 1st fundamental form is the metric g.) The ambient tangent bundle $T\tilde{M}|M \to M$ splits

$$T\tilde{M}|M = TM \oplus NM$$

into the orthogonal direct sum of the tangent bundle of M with the normal bundle $NM \to M$. Any section of $T\tilde{M}|M$ splits orthogonally into a direct sum of its tangential and normal part.
If $X, Y \in \mathcal{T}(M)$ are smooth vector fields on M, then e $\tilde{\nabla}_X Y$ is a well-defined section of $T\tilde{(M)}|M$. We know that $\nabla_X Y$ is the tangential component of e $\tilde{\nabla}_X Y$. If we write $II(X,Y)$ for the normal component then the orthogonal splitting of $\tilde{\nabla}_X Y$ has the form

$$\tilde{\nabla}_X Y = (\tilde{\nabla}_X Y)^T + (\tilde{\nabla}_X Y)^\perp = \nabla_X Y + II(X,Y) \quad (Gauss\ Formula) \tag{1.2}$$

where the normal component $II(X,Y) \in NM$ is called the *second fundamental form.* Equivalently,

$$II(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y$$

is the difference between the extrinsic $\tilde{\nabla}$ and the intrinsic connection ∇.

Lemma 1.1.22. *Let $M \subset \tilde{M}$ be a Riemannian submanifold and $II : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{N}(M)$ its second fundamental form.*

(1) II is $C^\infty(M)$-bilinear and symmetric.

(2) If $X, Y \in \mathcal{T}(M)$ are vector fields and $N \in \mathcal{N}(M)$ a normal field on M then

$$\left\langle \tilde{\nabla}_X N, Y \right\rangle = -\langle N, II(X,Y)\rangle \quad (Weingarten\ equation) \tag{1.3}$$

(3) If $X, Y, Z, W \in \mathcal{T}(M)$ are vector fields on M then

$$\begin{aligned} \widetilde{Rm}(X,Y,Z,W) = R(X,Y,Z,W) - \langle II(X,W), II(Y,Z)\rangle + \\ + \langle II(X,Z), II(Y,W)\rangle \quad (Gauss\ equation) \end{aligned} \tag{1.4}$$

(4) Let $g : I \to M$ be a curve in M and V a vector field in M along g. Then

$$\tilde{D}_t V = D_t V + II(\dot{g}, V) \quad (Gauss\ formula\ along\ a\ curve) \tag{1.5}$$

Proof. See Lemma 8.1, Lemma 8.3, Theorem 8.4, Lemma 8.5 in pages 134-138 of [31]. □

Remark 1.1.23. If we apply the Gauss formula along a curve to the special case where $V = \dot{g}$ is the velocity field then we get that

$$\tilde{D}_t\dot{g} = D_t\dot{g} + II(\dot{g}, \dot{g})$$

and we see that:

•If g is a geodesic in M, then $\tilde{D}_t\dot{g} = II(\dot{g}, \dot{g})$. Thus $II(V, V), V \in T_pM$, is the acceleration at p in $\tilde{M}$ of the geodesic g_V in M.

•If the curve g in M is a geodesic in $\tilde{M}$, then g is also a geodesic in M and $II(\dot{g}, \dot{g}) = 0$. Thus $II(V, V) = 0$ if the geodesic g_V in $\tilde{M}$, $V \in T_pM \subset T_p\tilde{M}$, happens to stay inside M. Since the second fundamental form is a symmetric bilinear form it is completely determined by its quadratic function $V \to II(V, V)$.

Gaussian curvature of codimension one Euclidean embeddings. We consider the simplest case of a Riemannian submanifold, namely that of an orientable hypersurface in Euclidean space, $M^n \subset R^{n+1}$. We shall associate curvature to the embedding. Choose a normal field N that is nonzero at every point of M. This is possible if M is orientable; in any case it is possible to choose such a normal field locally. Using N, we may write the second fundamental form $II : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{N}(M)$ as

$$II(X, Y) = h(X, Y)N \text{ so that } \langle\overline{\nabla}_XY, N\rangle = \langle II(X, Y), N\rangle = |N|^2h(X, Y)$$

where $h \in \mathcal{T}^2_0(M)$ is symmetric $(2, 0)$-tensor on M, the scalar second fundamental form. In this case of a codimension one embedding into Euclidean space $\mathbb{R}^{n+1}$ the Gauss and Weingarten formulas specialize to:

$$\overline{\nabla}_XY = \nabla_XY + h(X, Y)N \quad (Gauss\ formula) \tag{1.6}$$

$$|N|^2h(X, Y) = \langle\overline{\nabla}_XY, N\rangle = -\langle\overline{\nabla}_XN, Y\rangle \quad (Weingarten\ equation\ for\ N) \tag{1.7}$$

$$Rm(X, Y, Z, W) = |N|^2(h(X, W)h(Y, Z)-h(X, Z)h(Y, W)) \quad (Gauss\ equation) \tag{1.8}$$

Sectional curvature. Let (M, g) be a Riemannian manifold. We shall give a geometric interpretation of the Riemann curvature Rm tensor of M. Let p be a point of M. For each 2-dimensional subspace Π of the tangent space T_pM, let $S_\Pi \subset M$ be the surface in M that is the image under exp_p of Π, or rather the image of the part of Π where exp_p is a diffeomorphism, $S_\Pi = exp_p(\Pi \cap \epsilon_p)$. We give S_Π the induced metric so that $S_\Pi \subset M$ is a Riemannian embedding. Note that the tangent space of S_Π is $T_pS_\Pi = T_pexp_p(\Pi \cap \epsilon_p) = (exp_p)_*T_0\Pi = T_0\Pi = \Pi \subset T_pM$.

Definition 1.1.24. *Sectional curvature* at $p \in M$ is the function that to any tangent plane $\Pi \subset T_pM$ to the manifold associates the Gaussian curvature $K(\Pi) = K(S_\Pi)_p$ at p of the Riemannian surface S_π.

Theorem 1.1.25. *(**Gauss's Theorema Egregium**) Let $M \subset \mathbb{R}^n$ be a 2-dimensional submanifold and g the induced metric on M. For any $p \in M$ and any basis (v, w) for T_pM, the Gaussian curvature of M at p is given by*

$$K = \frac{Rm(v, w, w, v)}{|v|^2|w|^2 - \langle v, w\rangle^2}. \tag{1.9}$$

Therefore the Gaussian curvature is an isometry invariant of (M, g).

Proof. See page 143 of [31]. □

In particular, if the oriented Riemannian manifold $M^n \subset \mathbb{R}^{n+1}$ stands as an embedded codimension one manifold in $\mathbb{R}^{n+1}$ with Euclidean or Minkowski metric then the sectional curvature is

$$\begin{aligned} K(v, w) = |N|^2 \frac{h(v, v)h(w, w) - h(v, w)^2}{g(v, v)G(w, w) - G(v, w)^2} &= \\ = \frac{1}{|N|^2} \frac{\langle \overline{\nabla}_v, N\rangle\langle \overline{\nabla}_w, N\rangle - \langle \overline{\nabla}_v w, N\rangle^2}{\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2} \end{aligned} \tag{1.10}$$

where N is a nowhere zero normal field.

Proposition 1.1.26. *The sectional curvatures determine the Riemann curvature tensor of M.*

Proof. See page 37 of [27]. □

1.1.3 Isometries

If (M, g) and $(N, \tilde{g})$ are Riemannian manifolds, a diffeomorphism $f : M \to N$ is called an isometry if $f \circ \tilde{g} = g$. We say (M, g) and $(N, \tilde{g})$ are isometric if there exists an isometry between them. It is easy to verify that being isometric is an equivalence relation on the class of Riemannian manifolds. Riemannian geometry is concerned primarily with properties that are preserved by isometries. An isometry $f : (M, g) \to (M, g)$ is called an isometry of M. A composition of isometries and the inverse of an isometry are again isometries, so the set of isometries of M is a group, called the isometry group of M; it is denoted by $\mathcal{I}(M)$.
We will denote by d_xf the differential of a mapping f in a point x of M. We will denote by $\langle .|.\rangle_x$ the scalar product defined on the tangent space T_xM,

and by ds_x^2 the quadratic form associated to it. We recall that a mapping f is an isometry if it satisfies the following condition:

$$\langle d_x f(v) | d_x f(w) \rangle_{f(x)} = \langle v | w \rangle_x \ \ \forall x \in M, v, w \in T_x M.$$

Proposition 1.1.27. *Let M and N be two Riemannian manifolds of the same dimension. Assume M is connected and let*

$$f_1 : M \to f_1(M) \subseteq N, f_2 : M \to f_2(M) \subseteq N$$

be local isometries onto their range. If for some $y \in M$ we have $f_1(y) = f_2(y)$ and $d_y f_1 = d_y f_2$ then $f_1 = f_2$.
The conclusion holds in particular if f_1 and f_2 are isometries of M onto N.

Proof. See page 4 of [11]. □

1.1.4 Homogeneous and isotropic manifolds

Definition 1.1.28. M is a *homogeneous Riemannian manifold* if it admits a Lie group acting smoothly and transitively by isometries.

Definition 1.1.29. Given a point $x \in M$, M is *isotropic at p* if there exists a Lie group G acting smoothly on M by isometries such that the isotropy subgroup $G_x \subset G$ (the subgroup of elements of G that fix x) acts transitively on the set of unit vectors in $T_x M$, where $g \in G_x$ acts on $T_x M$ by $g_* : T_x M \to T_x M$.
A homogeneous Riemannian manifold that is isotropic at one point is isotropic at every point; in that case, M is said to be *homogeneous and isotropic.*

A homogeneous Riemannian manifold looks geometrically the same at every point, while an isotropic one looks the same in every direction.

1.1.5 Hints of Conformal Geometry: Circular Inversions

In this section we will give some important results of conformal geometry in $\mathbb{R}^n$. Conformal geometry in $\mathbb{R}^n$ allows us to calculate the isometries of $\mathbb{H}_r^n$ in the disc. We will prove that every isometry with respect to the hyperbolic structure in $\mathbb{D}_r^n$ is a conformal automorphism with respect to the Euclidian structure naturally defined by the immersion in $\mathbb{R}^n$, and viceversa. We will also see that there is an equivalence between $\mathbb{R}^n$ and the sphere $S_r^n \subset \mathbb{R}^{n+1}$ in the sense we will show in the sequel.

Definition 1.1.30. Two metrics g_1 and g_2 on a manifold M are said to be *conformal* to each other if there is a positive function $f \in C^\infty(M)$ such that $g_2 = fg_1$.

Definition 1.1.31. Let M and N be two Riemannian manifolds. We will say a diffeomorphism $f : M \to N$ is *conformal* if there exists a differentiable positive function a in M such that

$$\langle d_x f(v) | d_x f(w) \rangle_{f(x)} = a(x) \langle v|w \rangle_x \ \ \forall x \in M, \ v, w \in T_x M$$

i.e. the function f preserves angles but not necessary length. Manifolds endowed with a conformal structure are manifolds in which the angle between two vectors is defined.

We will denote by $\mathrm{Conf}(M,N)$ the set of conformal diffeomorphisms of M onto N. When $M = N$ we will only write $\mathrm{Conf}(M)$ and when the orientation is preserved we will write $\mathrm{Conf}^+(M)$.

Definition 1.1.32. Let $x_0 \in \mathbb{R}^n$ and $r > 0$. We will call *inversion* with respect to the sphere $M(x_0, r)$ of centre x_0 and radius $\sqrt{r}$ the mapping:

$$i_{x_0,r} : x \mapsto r \cdot \frac{x - x_0}{||x - x_0||^2} + x_0.$$

For us $i_{x_0,r}$ will be both a mapping of $\mathbb{R}^n \setminus \{x_0\}$ onto itself and a mapping $\mathbb{R}^n \cup \{\infty\}$ onto iself, where $\mathbb{R}^n \cup \{\infty\} \cong S^n$ is the one-point compactification of $\mathbb{R}^n$, and $i_{x_0,r}$ exchanges x_0 and ∞. In this section S^n_r will be endowed with its natural conformal structure and every open subset of $\mathbb{R}^n$ will be endowed with the conformal structure induced from $\mathbb{R}^n$.
We will say that two hyperplanes H_1 and H_2 in $\mathbb{R}^n$ are *orthogonal* if the lines $H_1^\perp$ and $H_2^\perp$ are orthogonal; consequently we will say that two intersecting spheres are orthogonal if for any point of their intersection the two tangent hyperplanes are orthogonal in the above sense; i.e. if x_0 and x_1 are the centres of the spheres, for each point x of the intersection $\langle x - x_0 | x - x_1 \rangle = 0$. In the following proposition we will list some important properties of inversions.

Proposition 1.1.33. *(1) $i_{x_0,r} \circ i_{x_0,s}$ is the dilation centred at x_0 of ratio $\frac{r}{s}$.*
(2) $i_{x_0,r}$ is a C^∞ involution of both $\mathbb{R}^n$ and S^n_r.
(3) $i_{x_0,r}|_{M(x_0,r)} = id$.
(4) $i_{x_0,r}$ is a conformal mapping.
(5) Given $r, s > 0$ and $x_1 \neq x_0$ the following facts are equivalent:
i) $M(x_1, s)$ is $i_{x_0,r}$-invariant;
ii) $M(x_0, r)$ is $i_{x_1,s}$-invariant;

iii) $||x_1 - x_0||^2 = r + s$*;*

iv) $M(x_1, s)$ *and* $M(x_0, r)$ *are orthogonal spheres.*

(6) Let $i = i_{x_0,r}$*; then*

i) H *hyperplane,* $H \ni x_0 \Rightarrow i(H) = H$*;*

ii) H *hyperplane,* $H \not\ni x_0 \Rightarrow i(H) = H$ *sphere,* $i(H) \ni x_0$*;*

iii) M *sphere,* $M \ni x_0 \Rightarrow i(M)$ *hyperplane,* $i(M) \not\ni x_0$*;*

iv) H *sphere,* $M \not\ni x_0 \Rightarrow i(M)$ *sphere,* $i(M) \not\ni x_0$*;*

v) i *operates bijectively on the set of all open balls and all open half-spaces in* $\mathbb{R}^n$*.*

Proof. See page 8 of [11]. □

For $n \geq 2$ we will deal with the set of all conformal diffeomorphisms between two domains of $\mathbb{R}^n$. The technique is completely different for the case $n = 2$ and the case $n \geq 3$; however, for the particular open sets we are interested in, the result is the same for all integers n.

We will give some important results only on the case $n = 2$. The reader can find the analogous results for $n \geq 3$ in [11].

A connected oriented Riemannian surface M admits a complex structure, and this structure is uniquely determined by the requirement that

$$f : \mathbb{C} \supset U \to M$$

is a holomorphic chart if and only if it preserves orientation and

$$ds^2_{f(z)}\big(d_z f(x)\big) = t(z) \cdot |w|^2 \quad \forall z \in U, w \in \mathbb{C}$$

for some function $t > 0$.
By the following proposition conformal geometry in dimension 2 reduces to a problem in the theory of functions of one complex variable.

Proposition 1.1.34. *If* M *and* N *are connected oriented Riemannian surfaces (naturally endowed with complex structures), the set of all conformal diffeomorphisms of* M *onto* N *is the set of all holomorphisms and all anti-holomorphisms of* M *onto* N*.*

Proof. See page 10 of [11]. □

Let us consider the Riemann sphere $S^2_r = \mathbb{CP}^1$ naturally identified with the set $\mathbb{C} \cup \{\infty\}$ (where $\infty = 0^{-1}$). We define the two classes of mappings of

$\mathbb{CP}^1$ onto itself by:

$$\begin{aligned} &homographies: && z \mapsto \frac{az+b}{cz+d} \\ &anti-homographies: && z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d} \end{aligned}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ varies in $Gl(2, \mathbb{C})$.

The following theorem settles the two-dimensional conformal geometry for the most important domains. We will identify $\mathbb{R}^2$ with $\mathbb{C}$, in such a way that $\mathbb{R}^2$ and D_r^2 are open subsets of $\mathbb{CP}^1$. If F is a set of mappings we denote by $c(F)$ the set $\{(z \mapsto \overline{f(z)}) : f \in F\}$ and by $-F$ the set $\{(z \mapsto -f(z)) : f \in F\}$.

Theorem 1.1.35. *The group* $\mathrm{Conf}^+(S_r^2)$ *consists of all homographies, and the group* $\mathrm{Conf}(S_r^2)$ *consists of all homographies and anti-homographies. For* $M = \mathbb{R}^2, D_r^2$ *we have*

$$\begin{aligned} &\mathrm{Conf}^+(M) = \{f|_M : f \in \mathrm{Conf}^+(S_r^2), f(M) = M\} \\ &\mathrm{Conf}(M) = \{f|_M : f \in \mathrm{Conf}(S_r^2), f(M) = M\} \end{aligned}$$

In particular:

$$\begin{aligned} &\mathrm{Conf}^+(\mathbb{C}) = \{(z \mapsto az+b) : a, b \in \mathbb{C}, a \neq 0\} \\ &\mathrm{Conf}(\mathbb{C}) = \mathrm{Conf}^+(\mathbb{C}) \cup c(\mathrm{Conf}^+(\mathbb{C})) \\ &\mathrm{Conf}^+(D_r^2) = \left\{ \left(z \mapsto e^{i\theta} \cdot \frac{z-\alpha}{1-\alpha\bar{z}} \right) : \theta \in \mathbb{R}, \alpha \in D_r^2 \right\} \\ &\mathrm{Conf}(D_r^2) = \mathrm{Conf}^+(D_r^2) \cup c(\mathrm{Conf}^+(D_r^2)). \end{aligned}$$

Proof. See page 11 of [11]. $\square$

Proposition 1.1.36. *If we identify* $\mathbb{CP}^1$ *with* $\mathbb{R}^2 \cup \{\infty\}$ *then* $\mathrm{Conf}(\mathbb{CP}^1)$ *consists of all and only the mappings of the form*

$$x \mapsto \lambda A i(x) + v$$

where $\lambda > 0,\ A \in O(2), i$ *is either the identity or an inversion and* $v \in \mathbb{R}^2$.

Proof. See page 13 of [11]. $\square$

Remark 1.1.37. Since translations, dilations and elements of $SO(n)$ preserve orientation, while elements of $O(n) \backslash SO(n)$ and inversions with respect to spheres reverse orientation, we have that $Ai \in \mathrm{Conf}(D_r^2)$ belongs to $\mathrm{Conf}^+(D_r^2)$ if and only if

$$[A \in SO(2),\ i = identity] \text{ or } [A \in O(2) \backslash SO(2),\ i = inversion].$$

Theorem 1.1.38. $\mathrm{Conf}(D_r^2)$ *consists of all and only the mappings of the form* $x \mapsto Ai(x)$*, where* $A \in O(2)$ *and* i *is either the identity or an inversion with respect to a sphere orthogonal to* ∂D_r^2*.*

Proof. By 1.1.33 (4) and (5) every mapping of the form Ai belongs to $\mathrm{Conf}(D_r^2)$. As for the converse, we remark that the set of all the mappings of the required form is a group: since the conjugation and all rotations belongs to $O(2)$ we only have to check that for $\alpha \in D_r^2$ the function $z \mapsto \frac{(\bar{z}-a)}{(1-\bar{a}\bar{z})}$ can be written as Ai. The sphere of centre $\frac{1}{\alpha}$ and squared radius $\frac{1}{|a|^2} - 1$ is orthogonal to ∂D_r^2; let i denote the inversion with respect to it; we have

$$\begin{aligned} i(z) &= \frac{1}{\alpha} + \Big(\frac{1}{|\alpha|^2} - 1\Big)\frac{1}{\bar{z} - \frac{1}{\bar{\alpha}}} = \\ &= \frac{1}{\alpha} + \frac{1-|\alpha|^2}{\alpha} \cdot \frac{1}{\bar{\alpha}\bar{z} - 1} = \frac{1}{\alpha} \cdot \frac{\bar{\alpha}\bar{z} - |\alpha|^2}{\bar{\alpha}\bar{z} - 1} = -\frac{\bar{\alpha}}{\alpha} \cdot \frac{\bar{z} - \alpha}{1 - \bar{\alpha}\bar{z}} \\ \Rightarrow & \frac{\bar{z} - \alpha}{1 - \bar{\alpha}\bar{z}} = -\frac{\alpha}{\bar{\alpha}} \cdot i(z). \end{aligned}$$

□

Theorem 1.1.39. $\mathcal{I}(\mathbb{D}_r^2) = \mathrm{Conf}(D_r^2)$, $\mathcal{I}^+(\mathbb{D}_r^2) = \mathrm{Conf}^+(D_r^2)$*. These groups operate transitively on* $\mathbb{D}_r^2$*.*

Proof. See page 22 of [11]. □

1.2 Models of the Euclidean Space $\mathbb{R}^n$, the Sphere S_r^n and the Hyperbolic space $\mathbb{H}_r^n$

Let n be a fixed natural number. In order to avoid trivialities we shall always assume $n \geq 2$. In every dimension $n \geq 2$ there exists only one complete, simply connected Riemannian manifold with sectional curvature 1, 0 or -1, up to isometries. These three manifolds are very important in Riemannian geometry because they are the fundamental model for constructing non simply connected manifolds with constant sectional curvature.
These three manifolds are the sphere S_r^n,the Euclidean space $\mathbb{R}^n$ and the hyperbolic space $\mathbb{H}_r^n$. Every complete manifold with constant curvature is universally covered by one of these spaces. In this thesis we will be working with complete manifolds with constant negative curvature $k < 0$ in the case of hyperbolic geometry, and with constant positive curvature $k > 0$ in the case of spherical geometry.
We will now introduce three classes of symmetric model spaces of Riemannian geometry: Euclidean space, spheres, and hyperbolic spaces.

1.2.1 The Euclidean Space $\mathbb{R}^n$

The Riemannian connection on $\mathbb{R}^n$ is the Euclidean connection:

$$\overline{\nabla}_X(Y^j\partial_j) = X(Y^j)\partial_j$$

Note that $\overline{\nabla}_{\partial_i}\partial_j = 0,\ 1 \le i, j \le n$. The derivative in the direction of tangent vector X of $Y = Y^i\partial_i$ is $\overline{\nabla}_X(Y) = X(Y^i)\partial_i$.
Now $R(\partial_i, \partial_j)\partial_k = 0$ as the basis vector fields commute, $[\partial_i, \partial_j] = 0$. Thus, the curvature tensor $Rm = 0$ so that the sectional curvature $K(\Pi) = 0$ for any plane $\Pi \subset T_p\mathbb{R}^n$ at any point $p \in \mathbb{R}^n$. This is obvious geometrically, since each plane section is actually a plane, which has zero Gaussian curvature.

The geodesics in the Euclidean space are the straight lines. They are the curves that minimize the arc length and are parametrized with constant velocity. The most important model Riemannian manifold is $\mathbb{R}^n$, with the Euclidean metric given $\bar{g}$ given by (1.1).

The isometry group of Euclidean geometry:

$$\mathcal{I}(\mathbb{R}^n) = \mathrm{Aff}(n) = \mathbb{R}^n \rtimes O(n)$$

acts transitively on $\mathbb{R}^n$ by the rule $(v, A)(x) = v + Ax$. The isotropy subgroup at $0 \in \mathbb{R}^n$ is $O(n)$ and the projection

$$O(n) \to \mathrm{Aff}(n) = O(T\mathbb{R}^n) \to \mathrm{Aff}(\mathbb{R}^n)/O(n) = \mathbb{R}^n$$

is the unit n-frame bundle of Euclidean geometry $\mathbb{R}^n$.

1.2.2 The Sphere S^n_r

Spherical geometry is the geometry of the two-dimensional surface of a sphere S^n_r. In Euclidean geometry, the basic concepts are points and lines. On a sphere, points are defined in the usual sense. The equivalents of lines in Euclidean geometry, are defined in the sense of the shortest paths between points, which are called *geodesics*. On a sphere, the geodesics are the great circles; other geometric concepts are defined as in plane geometry, but with straight lines replaced by great circles. Thus, in spherical geometry, angles are defined between great circles, defining the spherical trigonometry that differs from ordinary trigonometry in many respects; for instance, the sum of the interior angles of a triangle exceeds π.
The metric of the sphere S^n_r is the metric $\mathring{g}_r$, induced from the Euclidean metric on $\mathbb{R}^{n+1}$, called the round metric of radius r.

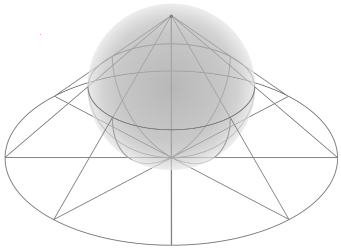

Figure 1.2: Illustration of a stereographic projection from the north pole onto a plane below the sphere.

We will use the technique of stereographic projection in order to reach our goals.

Isometries

We can immediately write a large group of isometries of S^n_r by observing that the linear action of the orthogonal group $O(n+1)$ on $\mathbb{R}^{n+1}$ preserves S^n_r and the Euclidean metric, so its restriction to S^n_r acts by isometries of the sphere.

Proposition 1.2.1. *$O(n+1)$ acts transitively on orthonormal bases on S^n_r. More precisely, given any two points $p, \tilde{p} \in S^n_r$, and orthonormal bases $\{E_i\}$ for $T_pS^n_r$ and $\{\tilde{E}_i\}$ for $T_{\tilde{p}}S^n_r$, there exists $f \in O(n+1)$ such that $f(p) = \tilde{p}$ and $f_*E_i = \tilde{E}_i$. In particular, S^n_r is homogeneous and isotropic.*

Proof. It suffices to show that given any $p \in S^n_r$ and any orthonormal basis $\{E_i\}$ for $T_pS^n_r$, there is an orthogonal map that takes the north pole $N = (0, ..., 0, r)$ to p and the standard basis $\{\partial_i\}$ for $T_NS^n_r$ to $\{E_i\}$. To do so, think of p as a vector of length r in $\mathbb{R}^{n+1}$, and let $\hat{p} = p/r$ denote the corresponding unit vector. Since the basis vectors $\{E_i\}$ are tangent to the sphere, they are orthogonal to $\hat{p}$, so $(E_1, ..., E_n, \hat{p})$ is an orthonormal basis for $\mathbb{R}^{n+1}$. Let α be the matrix whose columns are these basis vectors. Then $z \in O(n+1)$, and by elementary linear algebra z takes the standard basis vectors $(\partial_1, ..., \partial_{n+1})$ to $(E_1, ..., E_n, \tilde{p})$. In particular, $z(0, ..., 0, r) = p$. Moreover, since za acts linearly on $\mathbb{R}^{n+1}$, its push-forward is represented in standard coordinates by the same matrix, so $z_*\partial_i = E_i$ for $i = 1, ..., n$, and α is the desired orthogonal map. $\square$

Another important feature of the sphere is that it is locally conformally equivalent to the Euclidean space.

A conformal equivalence between $\mathbb{R}^n$ and the sphere $S^n_r \subset \mathbb{R}^{n+1}$ minus a point is provided by *stereographic projection* from the north pole. This is the map $s : S^n_r \setminus \{N\} \to \mathbb{R}^n$ that sends a point $P \in S^n_r \setminus \{N\} \subset \mathbb{R}^{n+1}$, written $P = (x^1_1, ..., x^n_1, y)$, to $x_2 \in \mathbb{R}^n$, where $U = (x^1_2, ..., x^n_2, 0)$ is the point where the line through N and P intersects the hyperplane $\{y = 0\}$ in $\mathbb{R}^{n+1}$ (Figure 1.6). Thus U is characterized by the fact that $\overrightarrow{NU} = \lambda\overrightarrow{NP}$ for some nonzero scalar λ. Writing $N = (0, r), U = (x_2, 0)$, and $P = (x_1, y) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, this leads to the system of equations

$$x_2^i = \lambda x_1^i, \quad -r = \lambda(y - r). \tag{1.11}$$

Solving the second equation for λ and inserting it into the first equation,we get the formula for stereographic projection

$$s(x_1, y) = x_2 = \frac{rx_1}{r - y}. \tag{1.12}$$

Clearly s is defined and smooth on all of $S_r^n \backslash \{N\}$. The easiest way to see that s is a diffeomorphism is to compute its inverse. Solving the two equations of (1.4) for y and x_1^i gives

$$x_1^i = \frac{x_2^i}{\lambda}, \quad y = r\frac{\lambda - 1}{\lambda}. \tag{1.13}$$

The point $P = s^{-1}(x_2)$ is characterized by these equations and the fact that P is on the sphere. Thus, substituting (1.6) into $|x_1|^2 + y^2 = r^2$ gives

$$\frac{|x_2|^2}{\lambda^2} + r^2\frac{(\lambda - 1)^2}{\lambda^2} = r^2,$$

from which we conclude

$$\lambda = \frac{|x_2|^2 + r^2}{2r^2}.$$

Inserting this back into (1.6) gives the formula

$$s^{-1}(x_2) = (x_1, y) = \Big(\frac{2r^2 x_2}{|x_2|^2 + r^2}, r\frac{|x_2|^2 - r^2}{|x_2|^2 + r^2}\Big), \tag{1.14}$$

which by construction maps $\mathbb{R}^n$ back to $S_r^n \setminus \{N\}$ and shows that s is a diffeormorphism.

Lemma 1.2.2. *Stereographic projection is a conformal equivalence between $S_r^n \setminus \{N\}$ and $\mathbb{R}^n$.*

Proof. The inverse map s^{-1} is a local parametrization, so we will use it to compute the pullback metric. Consider an arbitrary point $q \in \mathbb{R}^n$ and a vector $v \in T_q\mathbb{R}^n$, and compute

$$(s^{-1})^*\mathring{g}_r(v, v) = \mathring{g}_r(s_*^{-1}v, s_*^{-1}v) = \bar{g}(s_*^{-1}v, s_*^{-1}v),$$

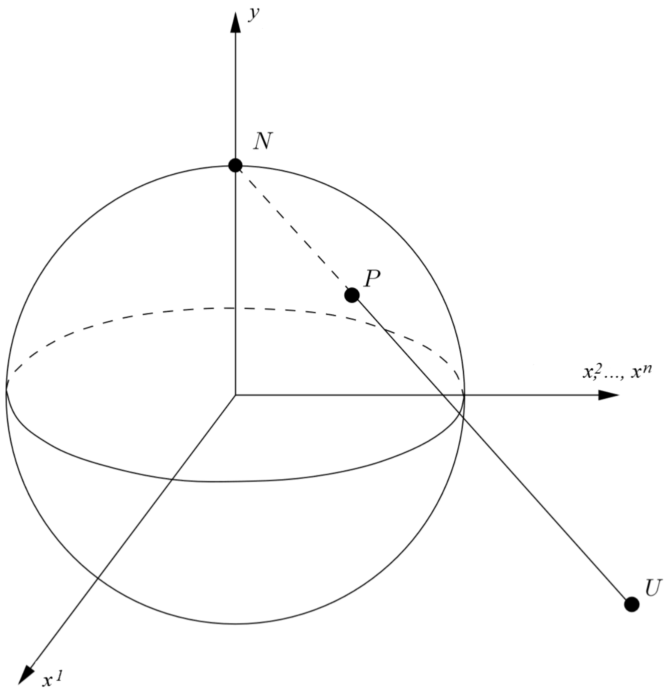

Figure 1.3: Stereographic Projection.

where $\bar{g}$ denotes the Euclidean metric on $\mathbb{R}^{n+1}$. Writing $v = v^i \partial_i$ and $s^{-1}(x_2) = (x_1(x_2), y(x_2))$, the usual formula for the push-forward of a vector can be written

$$\begin{aligned} s_*^{-1} v &= v^i \frac{\partial x^j}{\partial x_2^i} \frac{\partial}{\partial x_1^j} + v^i \frac{\partial y}{\partial x_2^i} \frac{\partial}{\partial y} \\ &= v x_1^j \frac{\partial}{\partial x_1^j} + v y \frac{\partial}{\partial y}. \end{aligned}$$

Now

$$
\begin{aligned}
vx_1^j &= v\Big(\frac{2r^2x_2^j}{|x_2|^2+r^2}\Big) \\
&= \frac{2r^2v^j}{|x_2|^2+r^2} - \frac{4r^2x_2^j\langle v, x_2\rangle}{(|x_2|^2+r^2)^2}; \\
vy &= v\Big(r\frac{|x_2|^2-r^2}{|x_2|^2+r^2}\Big) \\
&= \frac{2r\langle v, x_2\rangle}{|x_2|^2+r^2} - \frac{2r(|x_2|^2-r^2)\langle v, x_2\rangle}{(|x_2|^2+r^2)^2} \\
&= \frac{4r^3\langle v, x_2\rangle}{(|x_2|^2+r^2)^2},
\end{aligned}
$$

where we have used the notation $v(|x_2|^2) = 2\sum_k v^k x_2^k = 2\langle v, x_2\rangle$. Therefore,

$$
\begin{aligned}
\bar{g}(s_*^{-1}v, s_*^{-1}v) &= \sum_{j=1}^{n}(vx_1^j)^2 + (vy)^2 \\
&= \frac{4r^4|v|^2}{(|x_2|^2+r^2)^2} - \frac{16r^4\langle v, x_2\rangle^2}{(|x_2|^2+r^2)^3} + \frac{16r^4|x_2|^2\langle v, x_2\rangle^2}{(|x_2|^2+r^2)^4} \\
&\quad + \frac{16r^2\langle v, x_2\rangle^2}{(|x_2|^2+r^2)^4} \\
&= \frac{4r^4|v|^2}{(|x_2|^2+r^2)^2}.
\end{aligned}
$$

In other words,

$$
(s^{-1})^* \circ \mathring{g}_r = \frac{4r^4}{(|x_2|^2+r^2)^2}\delta_{ij}dv^i dv^j, \tag{1.15}
$$

where now $\delta_{ij}dv^idv^j = \bar{g}$ represents the Euclidean metric on $\mathbb{R}^n$, and so s is a conformal equivalence. □

An immediate consequence of this lemma is the fact that the sphere is *locally conformally flat*, i.e., every point $p \in S_r^n$ has a neighborhood that is conformally equivalent to an open set in $\mathbb{R}^n$. Stereographic projection gives such an equivalence for a neighborhood of any point except the north pole. We can get this equivalence for the north pole as well by rotating and then projecting stereographically or by projecting from the north pole.

Geodesics

The fact that the sphere is homogeneous and isotropic gives us a much easier way to determine the geodesics in all dimensions.

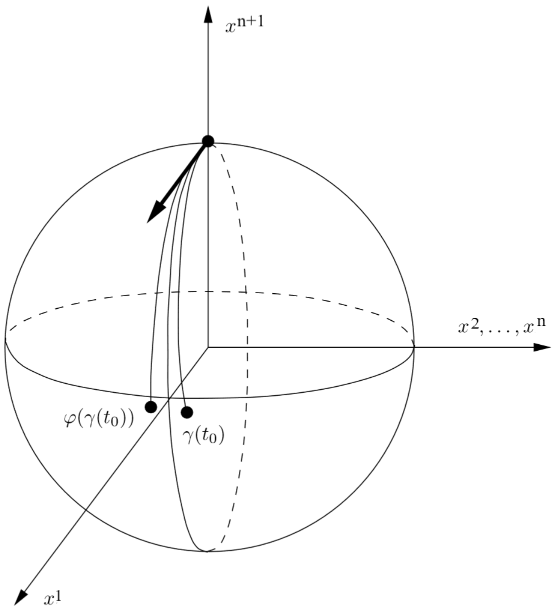

Figure 1.4: Identifying geodesics in S_r^2 by symmetry.

Proposition 1.2.3. *The geodesics on S_r^n are precisely the great circles, intersections of S_r^n with 2-planes through the origin, with constant speed parametrizations.*

Proof. First we consider a geodesic $\gamma(t) = (x^1(t), ..., x^{n+1}(t))$ starting at the north pole N whose initial velocity V is a multiple of ∂_1. It is intuitively evident by symmetry that this geodesic must remain along the meridian $x^2 = ... = x^n = 0$. To make this intuition rigorous, suppose not; that is, suppose there were a time t_0 such that $x^i(t_0) \neq 0$ for some $2 \leq i \leq n$. The linear map $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ sending x^i to $-x^i$ and leaving the other coordinates fixed is an isometry of the sphere that fixes $N = \gamma(0)$ and $V = \dot{\gamma}(0)$,and therefore it takes γ to γ. But $f(\gamma(t_0)) \neq \gamma(t_0)$, a contradiction. Since geodesics have constant speed, the geodesic with initial point N and initial velocity $c\partial_1$ must therefore be the circle where S_r^n intersects the (x^1, x^{n+1})-plane, with a constant speed parametrization. Since there is an orthogonal map taking any other initial point to N and any other initial vector to one of this form, and since orthogonal maps take planes through the origin to planes through the origin, it follows that the geodesics on S_r^n are precisely the intersections of S_r^n with 2-planes through the origin. $\square$

Curvature

We can now compute the sectional curvature of our homogeneous space. Note first that the sphere has an isometry group that acts transitively on orthonormal frames, and so acts transitively on 2-planes in the tangent bundle. Therefore it has *constant sectional curvature*, which means that the sectional curvature is the same for all planes at all points.

We need only to compute the sectional curvature for the plane Π spanned by (∂_1, ∂_2) at the north pole. The geodesics with initial velocity in Π are great circles in the (x^1, x^2, x^{n+1}) subspace. Therefore S_Π is isometric to the round 2-sphere of radius r embedded in $\mathbb{R}^3$. In spherical geometry, the tangent space $T_pS^n_r = p^\perp$ so that the normal space $N_pS^n_r = \mathbb{R}p$ is the line through p. Thus $N_p = \frac{1}{r}p$ is the outward pointing unit normal vector field. For any point $p \in S^n_r$ and any unit tangent vector $v \in T_pS^n_r$, Gauss formula for the velocity field of a curve (1.5) tells us that

$$II(v,v) = -\frac{1}{r}N \text{ and } h(v,v) = -\frac{1}{r}$$

as this is the acceleration of the geodesic through p with unit speed v , the great circle through p tangent to $v \in p^\perp$. Consider for instance the geodesic γ through $(0, ..., 0, r)$, $\gamma(t) = (r\sin(t/r), 0, ..., 0, r\cos(t/r))$, with unit speed $\dot\gamma(0) = (1, 0, ..., 0)$. Its acceleration is

$$\ddot\gamma(0) = -\frac{1}{r^2}\gamma(0) = -\frac{1}{r}(\frac{1}{r}\gamma(0)) = -\frac{1}{r}N(\gamma(0)).$$

Since $h(v,v) = -\frac{1}{r}$ for all unit tangent vectors v, $h(u,v) = 0$ for any orthonormal pair u, v of tangent vectors. Formula (1.10) tells us that the sectional curvature of S^n_r,

$$K(\Pi) = \frac{1}{r^2} \tag{1.16}$$

is constant.

Manifolds of Positive Curvature

Definition 1.2.4. The *diameter* of a Riemannian manifold is

$$diam(M) := sup\{d(p,q) : p, q \in M\}.$$

Theorem 1.2.5. ***(Bonnet's Theorem)*** *Let M be a complete, connected Riemannian manifold all of whose sectional curvatures are bounded below by a positive constant $1/r^2$. Then M is compact, with a finite fundamental group, and with diameter less than or equal to πr.*

Proof. See page 200 of [31]. □

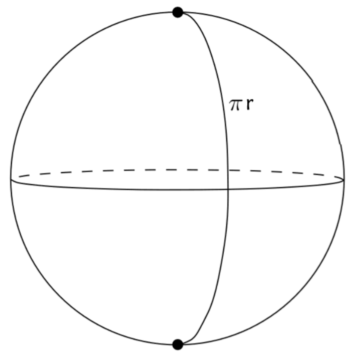

Figure 1.5: The diameter of S^n_r is πr.

1.2.3 The Hyperbolic Space $\mathbb{H}^n_r$

The third model space of Riemannian geometry is the hyperbolic space of dimension n. For each $r > 0$ we will describe a homogeneous, isotropic Riemannian manifold $\mathbb{H}^n_r$, called *hyperbolic space of radius r*. There are several models of the hyperbolic space. We will only work with two of them and in the sequel we will show that they are isometric.

Hyperboloid Model. In $\mathbb{R}^{n+1}$ let us consider the standard symmetric bilinear form of signature $(n, 1)$:

$$\langle x|y\rangle_{(n,1)} = \sum_{i=1}^{n} x^i \cdot y^i - x^{n+1} \cdot y_{n+1}.$$

A vector $x \in \mathbb{R}^{n+1}$ is *time-like, light-like* or *space-like* if $\langle x|x\rangle_{(n,1)}$ is negative, null, or positive respectively.
The hyperboloid model I^n_r is defined as follows:

$$I^n_r = \{x \in \mathbb{R}^{n+1} : \langle x|x\rangle_{(n,1)} = -r^2,\ x^n > 0\}$$

The set of points x such that $\langle x|x\rangle_{(n,1)} = -r^2$ is a hyperboloid of two sheets and the set I^n_r is the connected component with $x^n > 0$.
For us, a scalar product is a real non-degenerate symmetric bilinear form.

Corollary 1.2.6. *The hyperboloid I^n_r is a Riemannian manifold.*

Proof. Let $g_x(.,.)$ be any scalar product over $\mathbb{R}^n$. The function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = g_x(x, x)$ is smooth in every point and it's differential is: $df_x(y) = 2g_x(x, y)$. We have $g_x(x + y, x + y) = g_x(x, x) + 2g_x(x, y) + g_x(y, y)$. The component $g_x(x, y)$ is linear in y and $g_x(y, y) = o(||y||)$, where $||y||$ is

the usual Euclidean norm.
Since I_r^n is such that $\langle x, x\rangle_{(n,1)} = -1$, for every $x \in I_r^n$ the differential $y \mapsto 2g_x(x, y)$ is surjective. It follows that the set $\{x : f(x) + 1 = 0\}$ is a differential submanifold of R^{n+1} with codimension one.
For I_r^n, the tangent space $T_x I_r^n$ in x is precisely the kernel of the differential, thus the hyperplane orthogonal to x in the lorentzian scalar product:

$$T_x = \{y \in \mathbb{R}^{n+1} \mid \langle x, y\rangle_{(n,1)} = 0\} = \{x\}^{\perp}.$$

Since x is time-like, i.e.,$\langle x, x\rangle_{(n,1)} = -1$, the restriction of $\langle .|.\rangle_{(n,1)}$ to $\{x\}^{\perp}$ is positive definite, i.e. it is a scalar product on $\{x\}^{\perp}$. So, a metric is naturally defined on the tangent space to each point of I_r^n; it is easily verified that this metric is globally differentiable, and therefore I_r^n is endowed with a Riemannian structure. □

We shall denote by $\mathbb{I}_r^n$ the manifold I_r^n endowed with this structure.

The Disk Model. Let p be the restriction to $\mathbb{I}_r^n$ of the stereographic projection with respect to $(0, ..., 0, -1)$ of $\{x \in \mathbb{R}^{n+1} | x^{n+1} > 0\}$ onto $\mathbb{R}^n \times \{0\}$. We omit the last coordinate, so that the range of p is $\mathbb{R}^n$:

$$p(x) = \frac{(x^1, ..., x^n)}{1 + x^{n+1}}.$$

It is easily verified that p is a diffeomorphism of $\mathbb{I}_r^n$ onto the open Euclidean unit ball D_r^n of $\mathbb{R}^n$. The manifold D_r^n endowed with the pull-back metric with respect to p^{-1} will be denoted by $\mathbb{D}_r^n$. This manifold is canonically oriented as a domain of $\mathbb{R}^n$.

Proposition 1.2.7. *For any $r > 0$, the following Riemannian manifolds are mutually isometric.*

(a) The Hyperboloid Model $\mathbb{I}_r^n$ defined in coordinates $(x^1, ..., x^n, y)$ by the equation $y^2 - |x|^2 = -r^2$, with the metric

$$h_r^1 = \imath^* m,$$

where $\imath : \mathbb{I}_r^n \to \mathbb{R}^{n+1}$ is inclusion, and $m = (dx^1)^2 + ... + (dx^n)^2 - (dx^{n+1})^2$ is the Minkowski metric in $\mathbb{R}^{n+1}$ in coordinates $(x^1, ..., x^n, x^{n+1})$.

(b) The Poincaré Disk Model $\mathbb{D}_r^n$ of radius r in $\mathbb{R}^n$, defined in coordinates $(x^1, ..., x^n)$ with the metric

$$h_r^2 = 4r^4 \frac{(dx^1)^2 + ... + (dx^n)^2}{(r^2 - |x|^2)^2}.$$

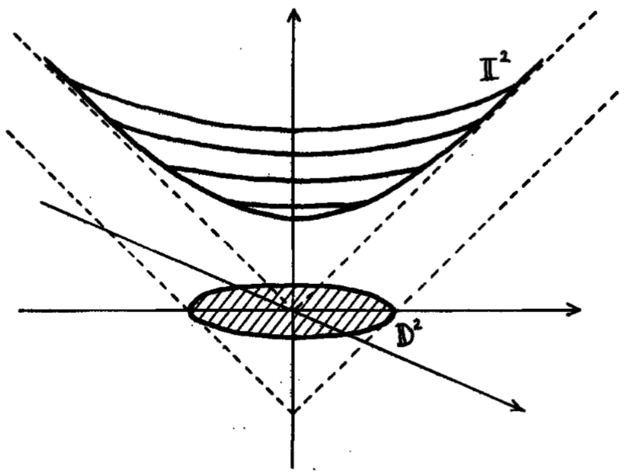

Figure 1.6: The hyperboloid and its projection onto the disk.

Proof. The proof will be given in the sequel as a proof for the theorem 1.2.10 □

Isometries
We will now use the results we have obtained from the conformal geometry in order to determine the group $\mathcal{I}(\mathbb{D}_r^n)$ of isometries of the disc D_r^n.
We will first introduce some elementary facts in linear algebra.

Let $\mathbb{R}^n$ be the n-dimensional real vector space and let $\langle.|.\rangle$ be a nondegenerate bi-linear form on $\mathbb{R}^n$. There exists a basis $\{v_1, ..., v_n\}$ of V such that

$$\langle v_i|v_j\rangle = \begin{cases} 0 & i \neq j \\ +1 & i = j \leq p \\ -1 & i = j > p. \end{cases}$$

If we set $q = n - p$ the pair (p, q) depends only on $\langle.|.\rangle$ and it is called the *signature*. If the linear group on $\mathbb{R}^n$ is denoted by $Gl(\mathbb{R}^n)$ we set

$$O(\mathbb{R}^n, \langle.|.\rangle) = \{A \in Gl(\mathbb{R}^n) : \langle Ax|Ay\rangle = \langle x|y\rangle \ \ \forall x, y \in \mathbb{R}^n\}.$$

If $\mathbb{R}^n = W \oplus W'$ and $p : \mathbb{R}^n \to W$ is the associated projection, we shall call *reflection* the linear mapping $r : v \mapsto 2p(v) - v$. It is easily verified that r is in $(\mathbb{R}^n, \langle.|.\rangle)$ if and only if $W' = W^{\perp}$, where $\perp$ denotes orthogonality with respect to $\langle.|.\rangle$. If this is the case we shall say that r is the reflection with respect to W, or parallel to $W^\perp$. From now on we shall call reflections only those with respect to a hyperplane, i.e. parallel to a vector such that $\langle v|v\rangle \neq 0$.

Proposition 1.2.8. *$O(\mathbb{R}^n, \langle .|.\rangle)$ is generated by reflections.*

Proof. We will prove it by induction on the dimension n. The first step is obvious. Assume the proposition is true for an integer n, consider now $\mathbb{R}^{n+1}$, let $\langle .|.\rangle$ be a non-degenerate bi-linear form on $\mathbb{R}^{n+1}$, let A belong to $O(\mathbb{R}^{n+1}, \langle .|.\rangle)$ and choose $v \in \mathbb{R}^{n+1}$ such that $\langle v|v\rangle \neq 0$. We can assume $\langle -Av - v, -Av - v\rangle \neq 0$: if this is not the case it is easi1y verified that $\langle Av - v, Av - v\rangle \neq 0$, hence we can replace A by $-A$; if $-A$ is a product of reflections then A is too. Let r be the reflection parallel to $Av - v$; since

$$v = \frac{1}{2}(Av + v) - \frac{1}{2}(Av - v) \quad \langle Av + v|Av - v\rangle = 0$$

then $r(v) = Av \Rightarrow (r \circ A)(v) = v \Rightarrow (r \circ A)|_{v^\perp} \in O(v^\perp, \langle .|.\rangle|_{v^\perp \times v^\perp})$. Since every reflection in $v^\perp$ extends to a reflection in $\mathbb{R}^{n+1}$ the induction hypothesis implies that A is a product of reflections. □

The symmetries of $\mathbb{H}^n_r$ are most easily seen in the hyperboloid model. Let $O(n, 1)$ denote the group of linear maps from $\mathbb{R}^{n+1}$ to itself that preserve the Minkowski metric. (This is called the *Lorentz group* in the physics literature.) Note that each element of $O(n, 1)$ preserves the set $\{y^2 - |x|^2 = r^2\}$, which has two components determined by $\{y > 0\}$ and $\{y < 0\}$. We let $O_+(n, 1)$ denote the subgroup of $O(n, 1)$ consisting of maps that take the component $\{y > 0\}$ to itself. Clearly $O_+(n, 1)$ preserves $\mathbb{H}^n_r$, and because it preserves m it acts on $\mathbb{H}^n_r$ as isometries.

Proposition 1.2.9. *$O_+(n, 1)$ acts transitively on the set of orthonormal bases on $\mathbb{H}^n_r$, and therefore $\mathbb{H}^n_r$ is homogeneous and isotropic.*

Proof. The proof is completely analogous to the proof of 1.2.1. Let $p \in \mathbb{H}^n_r$ and let $\{E_i\}$ be any orthonormal basis for $T_p\mathbb{H}^n_r$. We need to show that there is an orthogonal map that takes the vertex $N = (0, ..., 0, r)$ of the hyperboloid to p and the standard basis $\{\partial_i\}$ for $T_N\mathbb{H}^n_r$ to $\{E_i\}$. We think of p as a vector of length r in $\mathbb{R}^{n+1}$, and let $\hat{p} = p/r$ denote the corresponding unit vector. Since the basis vectors $\{E_i\}$ are tangent to the hyperboloid, they are orthogonal to $\hat{p}$, so $(E_1, ..., E_n, E_{n+1} = \hat{p})$ is an orthonormal basis for $\mathbb{R}^{n+1}$ such that m has the following expression in terms of the dual basis:

$$m = (f^1)^2 + ... + (f^n)^2 - (f^{n+1})^2.$$

Now, let z be the matrix whose columns are these basis vectors. Then $z \in O_+(n+1)$, and by elementary linear algebra z takes the standard basis vectors $(\partial_1, ..., \partial_{n+1})$ to $(E_1, ..., E_n, \tilde{p})$, and takes $N = (0, ..., 0, r)$ to p. Moreover, since α acts linearly on $\mathbb{R}^{n+1}$, its push-forward is represented in standard coordinates by the same matrix, so $z_* \partial_i = E_i$ for $i = 1, ..., n$, and z is the desired orthogonal map. $\square$

Using the theorem 1.1.39 the hyperbolic metric on $\mathbb{D}^n_r$ can be explicitely computed. Since $\mathbb{D}^n_r$ is an open subset of $\mathbb{R}^n$, its tangent bundle is canonically identified with $\mathbb{D}^n_r \times \mathbb{R}^n$.

Theorem 1.2.10. *For $x_2 \in \mathbb{D}^n_r$ and $v \in \mathbb{R}^n$ the metric is explicitely given by*

$$h^2_r(v, v) = \frac{4r^4|v|^2}{(r^2 - |x_2|^2)^2}.$$

Proof. Let us construct a diffeomorphism

$$s : \mathbb{I}^n_r \to \mathbb{D}^n_r$$

which we call *hyperbolic stereographic projection*, and which turns out to be an isometry between the two metrics given in (a) and (b) in 1.2.7.

Let $S \in \mathbb{R}^{n+1}$ denote the point $S = (0, ..., 0, -r)$. For any $P = (x^1_1, ..., x^n_1, y^{n+1}) \in \mathbb{I}^n_r \subset \mathbb{R}^{n+1}$, set $s(P) = x_2 \in \mathbb{D}^n_r$, where $U = (x_2, 0) \in \mathbb{R}^{n+1}$ is the point where the line through S and P intersects the hyperplane $y = 0$. U is characterized by $\overrightarrow{SU} = \lambda \overrightarrow{SP}$ for some $\lambda \neq 0$, or

$$x^i_2 = \lambda x^i_1, \; r = \lambda(y + r). \tag{1.17}$$

We have now

$$s(x_1, y) = x_2 = \frac{rx_1}{r + y}$$

and its inverse map

$$s^{-1}(x_2) = (x_1, y) = \left(\frac{2r^2 x_2}{r^2 - |x_2|^2}, r\frac{r^2 + |x_2|^2}{r^2 - |x_2|^2} \right).$$

We will show that $(s^{-1})^* h^1_r = h^2_r$. Let $q \in \mathbb{R}^n$ be an arbitrary point and let $v \in T_q \mathbb{D}^n_r$ and compute

$$(s^{-1})^* h^1_r(v, v) = h^1_r(s^{-1}_* v, s^{-1}_* v) = m(s^{-1}_* v, s^{-1}_* v).$$

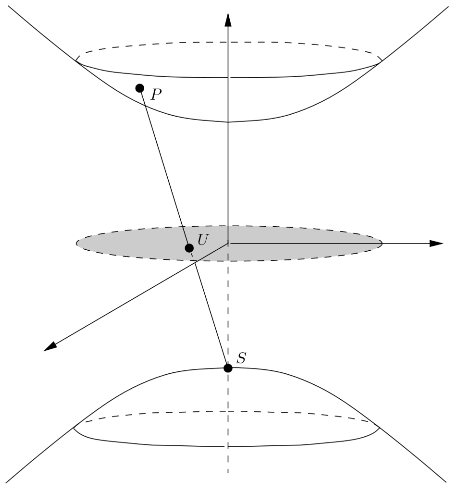

Figure 1.7: Hyperbolic stereographic projection.

Writing $v = v^j\partial_j$ and $s^{-1}(x_2) = (x_1(x_2), y(x_2))$ the usual formula for the pull-back of a vector can be written as:

$$\begin{aligned}(s^{-1})^* v &= v^j \frac{\partial x_1^j}{\partial x_2^j}\frac{\partial}{\partial x_1^j} + v^j \frac{\partial y}{\partial x_2^j}\frac{\partial}{\partial y} \\ &= v x_1^j \frac{\partial}{\partial x_1^j} + v y \frac{\partial}{\partial y}.\end{aligned}$$

The relevant equations are

$$\begin{aligned} v x_1^j &= \frac{2r^2 v^j}{r^2 - |x_2|^2} + \frac{4r^2 x_2^j \langle v, x_2\rangle}{(r^2-|x_2|^2)^2}; \\ vy &= \frac{r^3\langle v, x_2\rangle}{(r^2-|x_2|^2)^2}; \\ m(s_*^{-1}v, s_*^{-1}v) &= \sum_{j=1}^n (v x_1^j)^2 - (vy)^2 \\ &= \frac{4r^4|v|^2}{(r^2-|x_2|^2)^2} \\ &= h_r^2(v,v). \end{aligned}$$

where we have used the notation $v(|x_2|^2) = 2\sum_k v^k x_2^k = 2\langle v, x_2\rangle$.
In other words $(s^{-1})^* \circ h_r^1(v,v) = h_r^2(v,v)$. $\square$

This argument also shows that h_r^1 is positive definite, and thus is indeed a Riemannian metric, a fact that was not evident from the defining formula due to the fact that m is not positive definite.

Corollary 1.2.11. *At any point of $\mathbb{D}^n_r$ the hyperbolic metric is a positive multiple of the Euclidean one.*

Proof. See page 25 of [11]. $\square$

By this corollary the notion of conformal diffeomorphism on $\mathbb{D}^n_r$ is the same if we consider the Euclidean metric and the hyperbolic one. Using 1.1.39, this implies that all conformal diffeomorphisms of $\mathbb{D}^n_r$ with respect to the hyperbolic metric are isometries. This fact does not depend on the concrete representation of $\mathbb{H}^n_r$, so that we have the following proposition: *a sufficient condition for a diffeomorphism of $\mathbb{H}^n_r$ to preserve lengths is that it preserves angles.*

Geodesics

The geodesics of $\mathbb{H}^n_r$ are easily determined using the homogeneity and isotropy, as in the case of the sphere.

Proposition 1.2.12. *The geodesics on the hyperbolic spaces are the following curves, with constant speed parametrizations:*
Hyperboloid model: The "great hyperbolas", or intersections of $\mathbb{I}^n_r$ with 2-planes through the origin.
Poincaré disk model: The line segments through the origin and the circular arcs that intersect $\partial\mathbb{D}^n_r$ orthogonally.

Proof. We will only prove the case of the Poincaré disk model in dimension 2. Let us recall the hyperbolic stereographic projection $s : \mathbb{I}^n_r \to \mathbb{D}^n_r$ constructed before:

$$s(x_1, y) = x_2 = \frac{rx_1}{r+y},$$
$$s^{-1}(x_2) = (x_1, y) = \left(\frac{2r^2x_2}{r^2 - |x_2|^2}, r\frac{r^2 + |x_2|^2}{r^2 - |x_2|^2}\right).$$

A geodesic in the hyperboloid model is the set of points on $\mathbb{H}^2_r$ that solve a linear equation $\alpha_i x_1^i + \nu y = 0$, with a constant speed parametrization. In the special case $\nu = 0$, this hyperbola is mapped by s to a straight line segment through the origin, as can easily be seen from the geometric definition of s. If $\nu \neq 0$, we can divide through by $-\nu$ and write the linear equation as

$y = \alpha_i x_1^i = \langle \alpha, x_1 \rangle$ (for a different covector α). Under s^{-1}, this pulls back to the equation

$$r \frac{r^2 + |x_2|^2}{r^2 - |x_2|^2} = \frac{2r^2 \langle \alpha, x_2 \rangle}{r^2 - |x_2|^2}$$

on the disk, which implies to

$$|x_2|^2 - 2r\langle \alpha, x_2 \rangle + r^2 = 0.$$

Completing the square, we can write this as

$$|x_2 - r\alpha|^2 = r^2(|\alpha|^2 - 1). \tag{1.18}$$

If $|\alpha|^2 \leq 1$ this locus is either empty or a point on $\partial \mathbb{D}_r^2$, so it does not

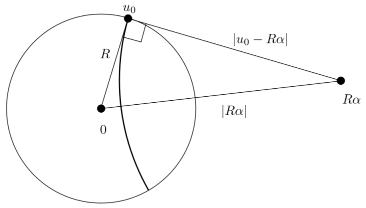

Figure 1.8: Geodesics are arcs of circles orthogonal to the boundary.

define a geodesic. When $|\alpha|^2 > 1$, this is the circle with center $r\alpha$ and radius $r\sqrt{|\alpha|^2 - 1}$. At a point u_0 where the circle intersects $\partial \mathbb{D}_r^2$, the three points $0, u_0$, and $r\alpha$ form a triangle with sides $|u_0| = r, |r\alpha|$, and $|u_0 - r\alpha|$ Figure 1.8, which satisfy the Pythagorean identity by (1.18); therefore the circle meets $\partial \mathbb{D}_r^2$ in a right angle. By the existence and uniqueness theorem, it is easy to see that the line segments through the origin and the circular arcs that intersect $\partial \mathbb{D}_r^2$ orthogonally are all the geodesics. In the higher-dimensional case, a geodesic on $\partial \mathbb{H}_r^n$ is determined by a 2-plane. If the 2-plane contains the point N, the corresponding geodesic on $\mathbb{D}_r^n$ is a line through the origin as before. Otherwise, we can conjugate with an orthogonal transformation in the $(x_1^1, ..., x_1^n)$ variables (which preserves h_r) to move this 2-plane so that it lies in the (x_1^1, x_1^{n+1}, y) subspace, and then we are in the same situation as in the 2-dimensional case. $\square$

Curvature

Finally we come to the hyperbolic spaces. It suffices to consider the point $N = (0, ..., r)$ in the hyperboloid model, and the plane $\Pi \subset T_N \mathbb{H}_r^n$ spanned by

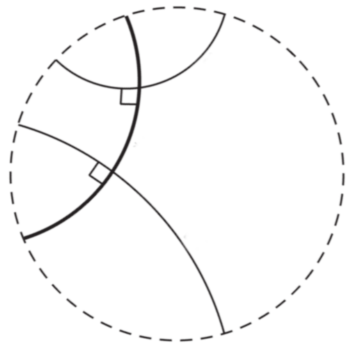

Figure 1.9: Geodesics in the Poincaré disk are arcs of circles.

$\partial/\partial x^1, \partial/\partial x^2$. The geodesics with initial velocities in Π are great hyperbolas lying in the (x^1, x^2, y) subspace; they sweep out a 2-dimensional hyperboloid that is easily seen to be isometric to $\mathbb{H}^2_r$. We will see that $K(\Pi) = -1/r^2$, so $\mathbb{H}^n_r$ has constant sectional curvature $-1/r^2$.
Before proving this in detail we give a result on the strict convexity of the distance function, which expresses qualitatively the fact that $\mathbb{H}^n_r$ is constantly negatively curved. If $x, y \in \mathbb{H}^n_r$ we shall denote by $\frac{(x+y)}{2}$ the middle point of the geodesic arc joining x and y.

Proposition 1.2.13. *Let γ_1, γ_2 be closed geodesic arcs in $\mathbb{H}^n_r$ having in common at most one endpoint and such that they are not arcs of the same maximal geodesic; let $x, x' \in \gamma_1$ and $y, y' \in \gamma_2$ with $x \neq x'$ and $y \neq y'$ and set $p = \frac{(x+x')}{2}$, $q = \frac{(y+y')}{2}$; then*

$$2d(p, q) < d(x, y) + d(x', y').$$

Proof. See page 37 of [11]. □

Remark 1.2.14. With the same symbols as above we have in $\mathbb{R}^n$ the same inequality

$$2d(p, q) \leq d(x, y) + d(x', y')$$

but we do have non-trivial cases when equality holds; moreover in S^n_r, endowed with the metric it inherits from $\mathbb{R}^{n+1}$, we have cases when the opposite inequality

$$2d(p, q) > d(x, y) + d(x', y')$$

holds. These facts express qualitatively the fact that $\mathbb{H}^n_r$, $\mathbb{R}^n$ and S^n_r have different curvature.

We will say three points in $\mathbb{H}^n_r$ are non-aligned if each geodesic in $\mathbb{H}^n_r$ contains at most two of them.

Corollary 1.2.15. *Let x, x', y be non-aligned points of $\mathbb{H}^n_r$ and define p as $\frac{(x+x')}{2}$; then*

$$2d(p, y) < d(x, y) + d(x', y).$$

Proof. See page 37 of [11]. $\square$

Lemma 1.2.16. *The sectional curvature of $\mathbb{H}^n_r$ in a point x with respect to a section $V \subset T_x\mathbb{H}^n_r$ is independent of V, x and n.*

Proof. Independence of V and x follows at once from the isometry-invariance of the curvature and from the fact that $\mathcal{I}(\mathbb{H}^n_r)$ operates transitively on the pairs x, V. As for the last assertion we only need to recall that by the fact that a p-dimensional hyperbolic subspace in $\mathbb{H}^n_r$ is isometrically diffeomorfic to $\mathbb{H}^p_r$ everything reduces to the case $n = 2$: in fact the image under the exponential mapping of a section at any point of $\mathbb{H}^n_r$ is a hyperbolic 2-subspace. $\square$

In hyperbolic geometry, the tangent space $T_pH^n_r = p^\perp$ so that the normal space $N_pH^n_r = \mathbb{R}_p$ is the line through p. Thus $N_p = \frac{1}{r}p$ is a normal vector field of constant square length $|N|^2 = -1$. For any point $p \in H^n_r$ and any unit tangent vector $v \in T_pH^n_r$, Gauss formula for the velocity field of a curve (1.5) tells us that

$$II(v, v) = \frac{1}{r}N \text{ and } h(v, v) = \frac{1}{r}$$

as this is the acceleration of the geodesic through p with unit speed v , the great hyperbola through p tangent to $v \in p^\perp$. Consider for instance the geodesic γ through $(0, ..., 0, r)$, $\gamma(t) = (r \sinh(t/r), 0, ..., 0, r \cosh(t/r))$, with unit speed $\dot{\gamma}(0) = (1, 0, ..., 0)$. Its acceleration is

$$\ddot{\gamma}(0) = \frac{1}{r^2}\gamma(0) = \frac{1}{r}(\frac{1}{r}\gamma(0)) = \frac{1}{r}N(\gamma(0)).$$

Since $h(v, v) = \frac{1}{r}$ for all unit tangent vectors v, $h(u, v) = 0$ for any orthonormal pair u, v of tangent vectors. Formula (1.10) tells us that the sectional curvature of $\mathbb{H}^n_r$,

$$K(\Pi) = -\frac{1}{r^2} \tag{1.19}$$

is constant.

Manifolds of Negative Curvature

The following theorem is a characterization of simply-connected manifolds of nonpositive sectional curvature.

Theorem 1.2.17. ***(The Cartan-Hadamard Theorem)*** *If M is a complete, connected manifold all of whose sectional curvatures are nonpositive, then for any point $p \in M, exp_p : T_pM \to M$ is a covering map. In particular, the universal covering space of M is diffeomorphic to $\mathbb{R}^n$. If M is simply connected, then M itself is diffeomorphic to $\mathbb{R}^n$.*

Proof. See page 196 of [31]. □

Lemma 1.2.18. *Suppose $\widetilde{M}$ and M are connected Riemannian manifolds, with $\widetilde{M}$ complete, and $l : \widetilde{M} \to M$ is a local isometry. Then M is complete and l is a covering map.*

Proof. See page 197 of [31]. □

1.2.4 Unique family of metrics which contains also $\mathbb{R}^n$: Manifolds of Constant Curvature

In this section we will prove a significant theorem relating to curvature and topology, which says that complete manifolds with constant sectional curvature are all quotients of the model spaces by discrete subgroups of their isometry. Before doing that we will introduce some definitions and state some of the most important local-global theorems of Riemannian geometry.

Theorem 1.2.19. ***(Uniqueness of Constant Curvature Metrics)*** *Let M be a complete, simply connected Riemannian n-manifold with constant sectional curvature K. Then M is isometric to one of the model spaces $\mathbb{R}^n$, S^n_r, or $\mathbb{H}^n_r$.*

Proof. We will consider the cases of positive and nonpositive sectional curvature separately. First suppose $K \leq 0$. Then theCartan-Hadamard theorem says that for any $p \in M$, $exp_p : T_pM \to M$ is a covering map. Since M is simply connected, it is a diffeomorphism. The pulled-back metric $\tilde{g} := exp_p^* g$, therefore is a globally defined metric on T_pM with constant sectional curvature K, and $exp_p : (T_pM, \tilde{g}) \to (M, g)$ is a global isometry. Moreover, since Euclidean coordinates for T_pM are normal coordinates for $\tilde{g}$, it must be given by one of the cases of formula (1.9); these in turn are globally isometric to $\mathbb{R}^n$ if $K = 0$ and $\mathbb{H}^n_r$ if $K = -\frac{1}{r^2}$.

In case $K = \frac{1}{r^2} > 0$, we have to proceed a little differently. Let $\{N, -N\}$ be the north and south poles in S^n_r, and observe that exp_N is a diffeomorphism from $D_{\pi r}(0) \subset T_pM$. If we choose any linear isometry

$z : T_N S_r^n \to T_pM$, then $(exp_p \circ z)^* g$ and $exp_N^* \mathring{g}_r$ are both metrics of constant curvature $\frac{1}{r^2}$ on $D_r(0) \subset T_N S_r^n$, and Euclidean coordinates on $T_N S_r^n$ are normal coordinates for both. Therefore, **??** shows that they are equal, so the map $Z : S_r^n \setminus \{-N\} \to M$ given by $Z = exp_p \circ z \circ exp_N^{-1}$ is a local isometry.

Now choose any point $Q \in S_r^n$ other than N or $-N$, and let $q = Z(Q) \in M$. Using the isometry $\tilde{z} = Z_* : T_Q S_r^n \to T_q M$, we can construct a similar map $\tilde{Z} = exp_q \circ \tilde{Z} \circ exp_Q^{-1} : S_r^n \setminus \{-Q\} \to M$, and the same argument shows that $\tilde{Z}$ is a local isometry. Because $Z(Q) = \tilde{Z}(Q)$ and $Z_* = \tilde{Z}_*$ at Q by construction, Z and $\tilde{Z}$ must agree where they overlap. Putting them together, therefore, we get a globally defined local isometry $F : S_r^n \to M$. After noting that M is compact by Bonnet's theorem and by the fact that any local diffeomorphism between compact, connected manifolds is a covering map, we complete the proof. □

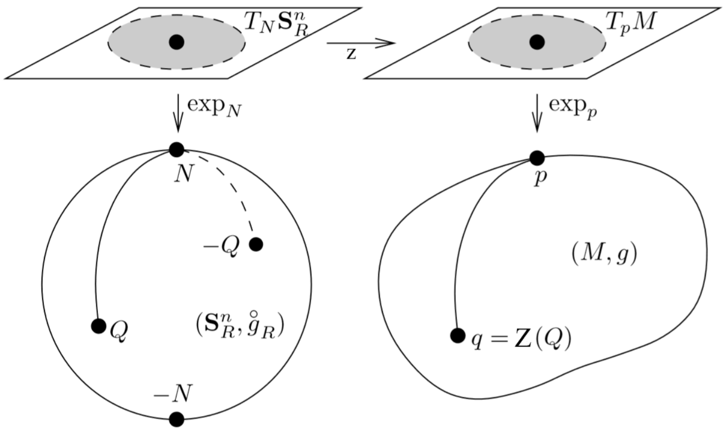

Figure 1.10: Constructing isometries when $K > 0$.

Corollary 1.2.20. ***(Classification of Constant Curvature Metrics)*** *Suppose M is a complete, connected Riemannian manifold with constant sectional curvature. Then M is isometric to $\widetilde{M}/\Gamma$, where $\widetilde{M}$ is one of the constant curvature model spaces $\mathbb{R}^n$, S_r^n, or $\mathbb{H}_r^n$, and Γ is a discrete subgroup of $(\widetilde{M})$, isomorphic to $\pi_1(M)$, and acting freely and properly discontinuously on $\widetilde{M}$.*

Proof. If $\pi : \widetilde{M} \to M$ is the universal covering space of M with the lifted metric $\tilde{g} = \pi^* g$, the precdeing theorem shows that $(\widetilde{M}, \tilde{g})$ is isometric to one of the model spaces. From covering space theory it follows that the group Γ of covering transformations is isomorphic to $\pi_1(M)$ and acts freely and properly discontinuosly on $\widetilde{M}$, and M is diffeomorphic to the quotient

$\widetilde{M}/\Gamma$. Moreover, if z is any covering transformation, $\pi \circ z = \pi$, and so $z^*\tilde{g} = z^*\pi^*g = \pi^*g = \tilde{g}$, so Γ acts by isometries. Finally, suppose $\{z_i\} \subset \Gamma$ is an infinite set with an accumulation point in $\mathcal{I}(\widetilde{M})$. Since the action on Γ is fixed-point free, for any point $\tilde{p} \in \widetilde{M}$ the set $\{z_i(\tilde{p})\}$ is infinite, and by continuity of the action it has an accumulation point in $\widetilde{M}$. But, this is impossible, since the points $\{z_i(\tilde{p})\}$ all project to the same point in M, and so form a discrete set. Thus Γ is dicrete in $\mathcal{I}(\widetilde{M})$. $\square$

A complete, connected Riemannian manifold with constant sectional curvature is called a *space form*. This result essentially reduces the classification of space forms to group theory.

In the spherical case the proof is connected with the representation theory of finite groups. In 2 dimensions the only examples are the sphere and the projective plane, meanwhile in dimension 3 there are many examples.

In the Euclidean case the classification of the space forms is known only in low dimensions. The fundamental groups of compact Euclidean space forms are examples of *crystallographic groups*, which are discrete groups of Euclidean isometries with compact quotients.

Chapter 2

Triangle Groups in the Euclidean, Spherical and Hyperbolic Geometry

Let X be either $\mathbb{R}^2$ or S^2_r or $\mathbb{H}^2_r$. Let $\tau = A_1A_2A_3$ be a triangle in X with angles $\pi/n_1, \pi/n_2, \pi/n_3$ and sides A_1, A_2, A_3 and r_1, r_2, r_3 be the reflection across the sides A_1, A_2, A_3 resepectively. Then the following relation hold

$$r_1^2 = r_2^2 = r_3^2 = 1$$

$$(r_1r_2)^{n_3} = (r_2r_3)^{n_1} = (r_3r_1)^{n_2} = 1$$

The triangle in X is defined using the following presentation

$$\tau = \{r_1, r_2, r_3 : r_1^2 = r_2^2 = r_3^2 = (r_1r_2)^{n_3} = (r_2r_3)^{n_1} = (r_3r_1)^{n_2} = 1\}$$

Definition 2.0.1. Let n_1, n_2, n_3 be the positive integers ≥ 2 and define δ to be

$$\delta = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} - 1. \tag{2.1}$$

So, a triangle group is a reflection group which is generated by the reflections of all sides of the triangle with angles $\pi/n_1, \pi/n_2, \pi/n_3$.
Now if

$$\delta > 0 \; \textit{then } \tau \textit{ is spherical},$$
$$\delta < 0 \; \textit{then } \tau \textit{ is hyperbolic},$$
$$\delta = 0 \; \textit{then } \tau \textit{ is euclidean}.$$

2.1 Trigonometry of the Sphere

If three points on any surface are joined by the shortest lines lying in the surface that it is possible to draw between these points a triangle is formed. Every such triangle has six parts, three sides and three angles. In general the sides are not straight lines but geodesic lines, i.e., the shortest lines that can be drawn on the surface connecting the points. Thus every class of surfaces gives rise to a special trigonometry whose object is the investigation of the relations between the parts of the triangle and the study of the functions necessary for the determination of the unknown parts of a triangle from a sufficient number of given parts. If the surface under consideration is the plane, the geodesies are straight lines and the triangles plane triangles, whose properties and those of the functions necessary for their solution are considered in plane trigonometry. If the points lie on the surface of a sphere the geodesies are arcs of great circles, the triangles are called spherical triangles, and the corresponding trigonometry, spherical trigonometry.
Spherical Trigonometry deals with the relations among the six parts of a spherical triangle and the problems which may be solved by means of these relations. The most important of these problems consist in the computation of the unknown parts of a spherical triangle when three parts are given.
The triangles formed by three points on the Earth's surface are not plane triangles but spherical triangles, thus the distances between are measured not along straight lines but along arcs of great circles.

Classification of Spherical Triangles. Spherical triangles are classified in two ways: first, with reference to the sides and second, with reference to the angles. A spherical triangle is said to be *equilateral, isosceles*, or *scalene*, according as it has three, two, or no equal sides.
A *right spherical triangle* is one which has a right angle; an *oblique spherical triangle* is one which has none of its angles a right angle. Oblique spherical triangles are *obtuse* or *acute* according as they have or have not an obtuse angle. Since the sum of the angles of a spherical triangle may have any value between π and 3π and no single angle can exceed π, a spherical triangle may have two or even three right angles. If it has two right angles it is called *birectangular*, if three, *trirectangular*. For the same reason a spherical triangle may have two or even three obtuse angles.
If two points on a sphere are at the extremities of the same diameter any great circle passing through one of the points will pass also through the other. Two such points, therefore, cannot be the vertices of a spherical triangle, because the great circles connecting these points with any third point will coincide and the resulting figure will not be a triangle but a *lune*. See Chapter 1 in

[22].
The Six Cases of Spherical Triangles. We will show that the six parts of any spherical triangle are so related that when any three are given the other three can be found. The three given parts may be:
I. The three sides.
II. The three angles.
III. Two sides and the included angle.
IV. Two angles and the included side.
V. Two sides and the angle opposite one of them.
VI. Two angles and the side opposite one of them.
We will call *spherical excess* the amount by which the sum of the angles exceeds π. This definition defines the behavior of the sphere and its edges. We know that the length of the edges on a spherical triangle will be greater than the length of the edges on a corresponding planar triangle, since they are curved. This definition allows a spherical triangle to have more than one right angle. Let us recall the definition of the spherical coordinates in a

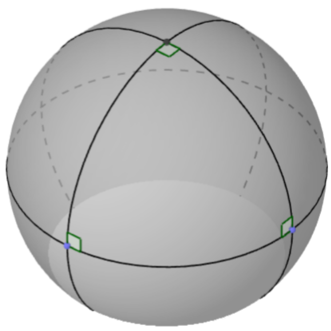

Figure 2.1: Spherical triangle with three right angles.

coordinate system in three dimentions. The *spherical coordinates* of a point P in the sphere of radius r are given by:

$$(x, y, z) = (r\cos(\theta)\sin(\phi), r\sin(\theta)\sin(\phi), r\cos(\phi))$$

where θ is the angle between the x-axis and the vector from the origin to the point P, and ϕ is the angle between the z-axis and the same vector.

Theorem 2.1.1. ***(The Spherical Pythagorean Theorem)*** *For a right triangle, τ on a sphere of radius r, with right angle π/n_3 and length $l_i > 0$ of the sides $e_i = A_jA_k$ is defined:*

$$\cos\left(\frac{l_3}{r}\right) = \cos\left(\frac{l_1}{r}\right) \cdot \cos\left(\frac{l_2}{r}\right) \tag{2.2}$$

Proof. See page 9 of [21]. □

There are two fundamental laws in the spherical trigonometry

Let $l_i \quad (i = 1, 2, 3)$ be the sides of a spherical triangle and let $\pi/n_1, \pi/n_2, \pi/n_3$ be its angles. Then:

Law of Cosines:

$$\cos\left(\frac{l_2}{r}\right) = \cos\left(\frac{l_1}{r}\right)\cos\left(\frac{l_2}{r}\right) + \sin\left(\frac{l_1}{r}\right)\sin\left(\frac{l_2}{r}\right)\cos(\pi/n_3). \tag{2.3}$$

Proof. See page 11 of [21]. □

Law of Sines:

$$\frac{\sin\left(\frac{l_1}{r}\right)}{\sin(\pi/n_1)} = \frac{\sin\left(\frac{l_2}{r}\right)}{\sin(\pi/n_2)} = \frac{\sin\left(\frac{l_3}{r}\right)}{\sin(\pi/n_3)}. \tag{2.4}$$

Proof. See page 14 of [21]. □

2.2 Trigonometry on the Hyperbolic Plane

Let τ be a compact hyperbolic triangle in the hyperbolic plane. As in the case for a Euclidean triangle, there are trigonometric laws in the hyperbolic plane relating the interior angles of τ and the hyperbolic lengths of the sides of τ, which we derive by linking the Euclidean and hyperbolic distances between a pair of points. As the measurement of angles of τ in hyperbolic geometry is the same as in the Euclidean one we may make use of the Euclidean trigonometric laws. There are also intrinsic ways of deriving the hyperbolic trigonometric laws that do not start from the Euclidean trigonometric laws. We first state some identities involving the hyperbolic trigonometric functions.

$$\cosh^2(x) - \sinh^2(x) = 1; \tag{2.5}$$

$$2\cosh(x)\sinh(x) = \sinh(2x); \tag{2.6}$$

$$\sinh^2(x) = \frac{1}{2}\cosh(2x) - \frac{1}{2}; \tag{2.7}$$

$$\cosh^2(x) = \frac{1}{2}\cosh(2x) + \frac{1}{2}; \tag{2.8}$$

$$\sinh^2(x)\cosh^2(y) + \cosh^2(x)\sinh^2(y) = \frac{1}{2}(\cosh(2x)\cosh(2y) - 1). \tag{2.9}$$

Let τ be a compact hyperbolic triangle in $\mathbb{D}^n_r$. Let $l_i > 0$ $(i = 1, 2, 3)$ be the hyperbolic lengths of its sides, and let $\pi/n_1, \pi/n_2, \pi/n_3$ be its interior angles. In the Hyperbolic space, there are three basic trigonometric laws, which we will state now. **Law of cosines I:**

$$\cosh(l_1) = \cosh(l_2)\cosh(l_3) - \sinh(l_3)\sinh(l_2)\cos(\pi/n_1). \tag{2.10}$$

Proof. See page 183 of [20]. □

Law of sines:

$$\frac{\sinh(l_1)}{\sin(\pi/n_1)} = \frac{\sinh(l_2)}{\sin(\pi/n_2)} = \frac{\sinh(l_3)}{\sin(\pi/n_3)}. \tag{2.11}$$

Proof. See page 183 of [20]. □

Law of cosines II:

$$\cos(\pi/n_3) = -\cos(\pi/n_1)\cos(\pi/n_2) + \sin(\pi/n_1)\sin(\pi/n_2)\cosh(l_3). \tag{2.12}$$

Proof. See page 183 of [20]. □

2.3 Tessellations

We consider tessellations of the euclidean, hyperbolic plane and of the two-sphere with tiles of triangle shape. We shall try to use them as tiles of a canonical tessellation by using the reflections in the sides of the triangles as generators of a group. In order to give a precise description of our approach, we start with the definition.

Definition 2.3.1. Let n_1, n_2, n_3 be integers ≥ 2, let

$$\delta = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} - 1, \tag{2.13}$$

and let τ be a triangle with angles $\pi/n_1, \pi/n_2, \pi/n_3$. If $\delta > 0$, τ is a spherical triangle. If $\delta = 0$, τ is euclidean, and if $\delta < 0$, τ is a hyperbolic triangle. Let l_1, l_2, l_3 be, respectively, the sides of τ opposite to the angles of size $\pi/n_1, \pi/n_2, \pi/n_3$, and let r_1, r_2, r_3, respectively, be the reflections of the particular two-dimensional space (sphere, euclidean, hyperbolic plane) in the straight lines (great circles in the spherical case) on which the sides r_1, r_2, r_3 lie. The group generated by r_1, r_2, r_3 shall be denoted by G_τ. The subgroup of G_τ consisting of words of even length in the generators r_1, r_2, r_3 shall be denoted by T_τ . The group T_τ consists of the orientation-preserving (i.e., proper) motions in G_τ and is therefore of index 2 in G_τ.
We call τ the *basic triangle* of the group G_τ.

We shall prove that the triangle τ is a canonical fundamental region for the groups G_τ in all cases. We shall also give defining relations for the generators of both the groups G_τ and T_τ, and, finally, we shall give matrix representations for the groups $T_\tau(2, n_2, n_3)$ by writing down explicitly 2×2 matrices for a set of generating fractional linear substitutions of T_τ. It is important to note that we have to read a product of motions (euclidean or noneuclidean) if represented by matrices, from right to left. Stated explicitly: *If r_2 and r_3 are matrices representing elements of the group T_τ, then $r_2 r_3$ represents the rigid motion which arises if first r_3 and then r_2 is carried out.* In most cases, this rule will not become apparent because it so happens that the defining relations of most of the groups which we shall encounter are symmetric expressions in the generators, meaning that they produce isomorphic (and not merely antiisomorphic) groups if read from left to right or from right to left. We will now give a detailed description of the possible triangle groups. *We shall frequently use the fact that the groups G_τ and T_τ are independent of the order in which n_1, n_2, n_3 are listed.*
The procedure outlined above for the construction of discontinuous groups from a tile which is assumed to be a canonical fundamental region of the

group poses two problems. First, we have to prove that the group generated by the motions which move the basic tile τ into an adjacent position has τ as a fundamental region. This means that the images of the basic tile under the repeated action of the generating motions will fill the two-dimensional space without gaps and overlapping. Second, we want to find a set of defining relations for the generators of our group.
Both problems split into a "local" and a "global" part. The *local tessellation* problem is the question: If we consider only those images of the original tile which have at least one point in common with it and which can be obtained by a sequence of generating motions which never move the original tile into a position completely disjoint from it, will these adjacent tiles cover a certain area completely enclosing the original one without gaps and overlapping? The *global tessellation* problem would be the question formulated above involving the whole two-dimensional space.
The problem of finding defining relations has to be preceded by the following remark: To every word in the generators of the group of a canonical tessellation, there belongs a chain of images of the fundamental region. This chain has properties which will be described below and is closed if and only if the word is a relator. In the case of the triangle groups, the generators are r_1, r_2, r_3. To describe this chain, we assign to every generator or its inverse a definite side of the canonical polygon. We call a side s of the polygon and an element g which is a generator or the inverse of a generator of the group associated if g maps the original tile (in the case of triangle groups: τ) onto an adjacent triangle $g\tau$ such that τ and $g\tau$ have s in common. In the case of the triangle groups, the generators coincide with their inverses; r_1, r_2, r_3 are respectively associated with l_1, l_2, l_3. If γ is any element of the groups, we can recognize the images of the sides of the fundamental tile in its image under the action of γ (even if some of the sides should be equal in length). Consider now an element

$$\gamma = g_1 g_2 \cdots g_{n-1} g_n,$$

where the g_ν ($\nu = 1, ..., n$) are generators or their inverses and where it never happens that a g_ν is followed by its inverse.

Lemma 2.3.2. *The images I_ν of the canonical fundamental region I_0, (i.e., of τ in the case of triangle groups G_τ) under the action of*

$$\gamma_\nu = g_1 g_2 \cdots g_\nu,$$

is adjacent to the image $I_{\nu+l}$ obtained under the action of

$$\gamma_{\nu+1} = g_1 g_2 \cdots g_\nu g_{\nu+1},$$

and the common side of I_ν and $I_{\nu+1}$ is the side of I_ν associated with $g_{\nu+l}$. This is also true for $\nu = 0$, if we denote the unit element by g_0.

Proof. The proof consists of the following remark: If I_ν is mapped onto the adjacent $I_{\nu+1}$ and if ${\gamma_\nu}^{-1}$ maps I_ν onto I_0, then $g_{\nu+l}$ maps I_0 onto its adjacent neighbor which has the side $s_{\nu+l}$ of I_0 in common with I_0, where $s_{\nu+1}$ is associated with $g_{\nu+1}$. Now, if $\gamma_\nu I_0 = I_\nu$, then $\gamma_\nu g_{\nu+1} I_0 = \gamma_{\nu+1} I_0$ has exactly the property used for the definition of $I_{\nu+1}$.
It follows that $\gamma = \gamma_n = 1$ if and only if I_n coincides with I_0, including the original labeling of the sides. The relations in the group of a canonical tessellation correspond therefore to the closed chains of images of the fundamental region. Now we define a local relation as a relation in which all images of the fundamental region I_0, (including I_0) have one point in common. All relations associated with chains whose members do not have one point in common are called *global relations*. The important fact which we wish to establish will always be that the local relations define the group. This is comparatively easy in the euclidean and the spherical cases, but it is more difficult in the hyperbolic case, even if lI_0 is merely a triangle τ.
The condition that the angles in a triangle τ be unit fractions of π guarantees that the local tessellation problem has an affirmative answer if τ is the basic tile. The local relations for the reflections r_1, r_2, r_3 in the sides of τ are established easily. □

A simple enumeration shows: The only (unordered) triplets n_1, n_2, n_3 of positive integers ≥ 2 for which the quantity δ of (2.13) satisfies the condition $\delta = 0$ are $(2,3,6), (2,4,4), (3,3,3)$ or $\delta > 0$ are $(2,2,n_3 \geq 2), (2,3,3), (2,3,4), (2,3,5)$.

2.3.1 Tessellations of the Euclidean plane

We shall deal with the euclidean case ($\delta = 0$) first and prove:

Theorem 2.3.3. *For $\delta = 0$, G_τ is defined by the local relations*

$$r_1^2 = r_2^2 = r_3^2 = 1, \tag{2.14}$$

$$(r_1 r_2)^{n_3} = (r_2 r_3)^{n_1} = (r_3 r_1)^{n_2} = 1. \tag{2.15}$$

The subgroup T_τ of index 2, consisting of orientation-preserving euclidean motions, is defined by two generators u, v which are rotations with two of the vertices of τ as center and the relations

$$u^{n_3} = v^{r_2} = (uv)^{r_1} = 1, \tag{2.16}$$

where $u = r_1 r_2$ and $v = r_3 r_1$.

Proof. Details will be given in the discussion of the individual cases.
Case 1. The group $T_\tau(2,3,6)$. We shall use the remark that we may permute ln_1, n_2, n_3 without changing T_τ and we shall prove: $T_\tau(6,3,2)$ is generated by the elements u and v, which, as euclidean motions, are repre- sented respectively by the substitutions

$$z' = ez \quad \text{and} \quad z' - 1 = \epsilon^2(z-1) \quad (\epsilon = e^{i\pi/3} = \frac{1}{2} + \frac{1}{2}i\sqrt{3}). \tag{2.17}$$

The relations

$$u^6 = v^3 = (vu)^2 = 1 \tag{2.18}$$

are defining relations for $T_\tau(2,3,6)$. The elements v_0, v_2 given by

$$v_0 = vu^4, \quad v_2 = u^2vu^2 \tag{2.19}$$

generate a free abelian normal subgroup A of T_τ which is of index 6 and has the elements u^ν $(\nu = 0, 1, ..., 5)$ as coset representatives. v_0 and v_2 are represented, respectively, by the translations

$$z' = z + 1 - \epsilon^2 \quad \text{and} \quad z' = z + \epsilon^2 - \epsilon^4. \tag{2.20}$$

A canonical fundamental region for $T_\tau(2,3,6)$ is given by the quadrilateral Q with vertices

$$0, \frac{1}{2}\sqrt{3}\epsilon, 1, \frac{1}{2}\sqrt{3}\epsilon^5.$$

The first three of these points are the vertices of a triangle τ with angles $\pi/6, \pi/2, \pi/3$, which is a canonical fundamental region for $G_\tau(6,2,3)$. Figure 2.2c gives τ (shaded) and part of the tesselation arising from τ under the action of G_τ. The quadrilateral Q arises from τ by its union with τ', which is the reflection of τ in the real axis.
We start the proof of these statements with an analysis of the abstract group presented by two generators u, v with defining relations (2.18). If we add the relation $vu^{-2} = 1$, this group is mapped onto the cyclic group of order 6 generated by u. The Reidemeister-Schreier method gives us as generators for the kernel of the mapping the elements

$$v_\mu = u^\mu v u^{4-\mu} \quad (\mu = 0, 1, ..., 5),$$

and the relations $v^3 = 1$ and $(vu)^2 = 1$ and their conjugates produce the relations

$$v_0v_2v_4 = v_lv_3v_5 = 1 \quad \text{and} \quad v_0v_3 = v_1v_4 = v_2v_5 = 1,$$

which enable us to express all the v_μ in terms of v_0 and v_2. In addition, they produce the relation $v_0v_2 = v_2v_0$.

That $T_\tau(2,3,6)$ is generated by the substitutions (2.17) is an elementary geometric remark, and we see that they satisfy the relations (2.18) for u and v. However, we have to show that they do not satisfy other relations which are not derivable from (2.18). This follows from the fact that the abstract group defined by u, v, and (2.18) has the following solution for the word problem: Every element has a unique expression

$$u^\nu v_0^k v_2^l \quad (\nu = 0, 1, ..., 5; \;\; k, l = 0, \pm 1, \pm 2, ...). \tag{2.21}$$

The corresponding substitution, arising from a replacement of u, v_0, v_2 by the rigid motions in (2.17) and (2.20), is

$$z' = \epsilon^\nu z + \epsilon^\nu [k(1-\epsilon^2) + l(\epsilon^2 - \epsilon^4)] \tag{2.22}$$

which is the identical substitution if and only if (2.21) is the unit element, represented by $\nu = k = l = 0$.

It remains to be shown that the quadrilateral Q defined above produces a covering of the plane without gaps and overlappings under the action of the motions (2.22). This can be done by observing first that the action of $u^\nu \;\; (\nu = 0, ..., 5)$ in Q produces a regular hexagon (see Figure 2.2 c, and then showing that for $\nu = 0$, the translations (2.22) move this hexagon so that its replicas form a tesselation of the plane. This task can be simplified by showing that the hexagon can be replaced by the parallelogram with the vertices

$$0, 1-\epsilon^2, \sqrt{3}i, 1-\epsilon^4, \;\; (\sqrt{3}i = \epsilon^2 - \epsilon^4).$$

Case 2. The group $T_\tau(3,3,3)$. This is a subgroup of index two in $T_\tau(2,3,6)$. Its canonical fundamental region is a rhombus with vertices

$$0, 1, \epsilon, \epsilon^2.$$

The generating elements u and v can be represented respectively by the rigid motions (rotations)

$$z' = \epsilon^2 z \;\text{ and }\; z' - \epsilon = \epsilon^2 (z - \epsilon).$$

The elements $v_0 = vu^{-l}$ and $v_1 = uvu^{-2}$ generate a free abelian subgroup A with cyclic quotient group and $1, u, u^2$ as coset representation in T_τ. The elements v_0 and v_1 are represented respectively by the translations

$$z' = z + \epsilon + l \;\text{ and }\; z' = z + \epsilon^2 - 1.$$

As a fundamental region for A, we can use the same hexagon as in the case of $T_\tau(6,3,2)$. The situation is illustrated by Figure 2.2 a.

Case 3. The group $T_\tau(2,4,4)$. The situation is illustrated by Figure 2.2b. It is so close to the chessboard tesselation that we need not go into the details. $\square$

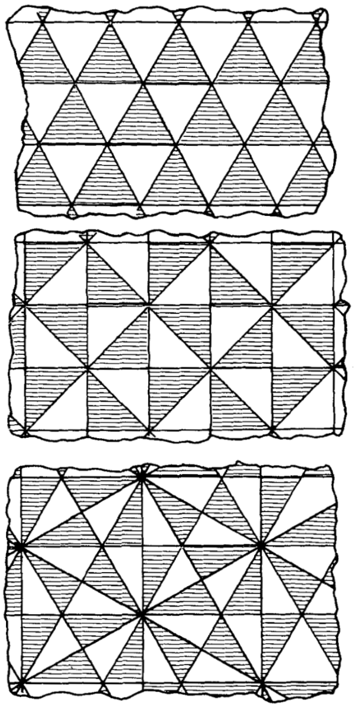

Figure 2.2: Tessellations of the Euclidean plane.

2.3.2 Tessellations of the Sphere

We will now consider the case where $\delta > 0$. The details of the structure of the groups generated by reflections r_1, r_2, r_3 in the sides of a spherical triangle will be described separately as cases $1-4$. As a summary, we state:

Theorem 2.3.4. *The reflections r_1, r_2, r_3 in the sides of a spherical triangle τ generate a group G_τ for which τ is a canonical fundamental region. The local relations (2.14), (2.15) define the group.*

Case 1. The *dihedral groups* $T_\tau(2,2,n_3)$. We choose as the original triangle τ on the sphere the triangle with one vertex at the south pole and with vertices $z = 1$ and $z = \epsilon$ on the equator, which is already in the z plane. Stereographic projection maps the south pole onto $z = 0$. Reflection of τ in the real axis produces the triangle τ', which, together with τ, forms a fundamental region of T_τ. The motions u and v can be defined respectively by the matrices

$$U = \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}, \quad V = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}; \quad U^n = V^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.23}$$

U defines a rotation with $z = 0$ as fixed point and with $2\pi/n_3$ as angle of rotation. This will be the generator u of T_τ. The generator v of T_τ, defined by V or, alternatively, by $z' = i/(iz)$, is of order 2 and represents a rotation of the sphere with $z = 1$ as a fixed point. We have

$$u_3^n = v^2 = (uv)^2 = 1, \tag{2.24}$$

where uv has the points $\pm\epsilon$ as fixed points. From (2.24), it follows that the group generated by u, v with defining relations (2.24) is of order $2n_3$, and that every one of its elements can be expressed uniquely in the form

$$u^\nu v^\delta \quad (\nu = 0, ..., n-1; \;\; \delta = 0, 1).$$

The corresponding motions of the sphere move $\tau \cup \tau'$ into $2n_3$ positions which cover the sphere without gaps and overlappings. The matrices U, V themselves generate a group of order $4n_3$ which is a nonsplitting central extension of $T_\tau(2, 2, n_3)$ and can be defined by the relations

$$U^{n_3} = V^2 = (UV)^2 \tag{2.25}$$

where $U^{n_3} = V^2$ belongs to the center and is of order 2. [To prove this remark, observe that $V^2 = (UV)^2$ can be written as

$$U^{-1} = VUV^{-1},$$

which implies

$$U^{-n_3} = VU^{n_3}V^{-1} = U^{n_3}.$$

Therefore $U^{2n_3} = V^4 = 1$ and the quotient group of the group generated by V^2 is T_τ.] This occurrence is typical for the representation of groups of spherical rotations in terms of unitary matrices. This group is *not* isomorphic with $G_\tau(2, 2, n_3)$, which is also of order $4n$ and which has, in the case of an even $n_3 = 2n_2$, a center element which is of order 2. But here this center element generates a direct factor of $G_\tau(2, 2, 4n_2)$. It is represented by the selfmapping of the sphere which maps every point into its diametric opposite and can be expressed as $r_3u^{n_2}$, where $u = r_1r_2, v = r_2r_3$. We can verify that in $G_\tau(2,2,2n_2)$, given by

$$r_1^2 = r_2^2 = r_3^2 = 1, \quad r_1r_2 = u, \quad r_2r_3 = v, u^{2n_2} = v^2 = (uv)^2 = 1, \qquad (2.26)$$

the element $r_3u^{n_2}$ belongs to the center. It is obviously not an element of the subgroup $T_\tau(2,2,2n_2)$ since it does not have an even length when expressed as a word in r_1, r_2, r_3. Also, $r_3u^{n_2}$ is of order 2 since

$$r_3u^{n_2}r_3u^{n_2} = (ur_3^{-1})^{n_2}u^{n_2} = v^{-1}u^{-n_2}vu^{n_2} = (v^{-1}uv)^{-n_2}u^{n_2}$$

and $v^{-1}uvu = v^{-2}(vu)^2 = 1$, which implies $v^{-1}uv = u^{-1}$. We may add that, in (2.26), r_1, r_2, r_3 are, respectively, the reflections of $\hat{C}$ in the real axis, in the line $z = \epsilon t(-\infty < t < \infty)$, and in the unit circle.

Case 2. The tetrahedral group $T_\tau(3,3,2)$*, and the group* $G_\tau(2,3,3)$. A tessellation of the sphere with triangles congruent to τ (which has angles $\pi/3, \pi/3, \pi/2$) arises if we inscribe a regular tetrahedron in a sphere and mark its vertices together with the projections of the centers of the faces and the midpoints of the edges on the sphere, the center of projection being the center of the sphere. Repeated reflection of the sphere in the sides of the triangle τ produces the group $G_\tau(3,3,2)$ with generators r_1, r_2, r_3 and defining relations

$$r_1^2 = r_2^2 = r_3^2 = 1, \quad (r_1r_2)^3 = (r_2r_3)^3 = (r_3r_1)^2 = 1. \qquad (2.27)$$

This group is isomorphic with $\sum_4$, the symmetric group of permutation of four symbols, as may be seen from the definition of this group (see pages 47-72 of [40]) as a quotient group of the braid group on four strings. We merely have to identify r_1, r_2, r_3, respectively, with $\sigma_l, \sigma_2, \sigma_3$ in Artin's notation.
$T_\tau(3,3,2)$ is the subgroup of $G_\tau(3,3,2)$ composed of words of even length in r_1, r_2, r_3. It is generated by $u = r_1r_2$ and $v = r_2r_3$ with defining relations

$$u^3 = v^3 = (uv)^2 = 1. \qquad (2.28)$$

This is the *tetrahedral group*, isomorphic with A_4, the alternating group of permutations of four symbols. It is represented as the group of proper motions (i.e., rotations) of the sphere which carry the original tetrahedron into itself. As its fundamental region, we may choose the union Q of τ and one of its reflected image τ'; the vertices of Q are now two vertices of the tetrahedron and the midpoint of a face on which both of these vertices lie. The midpoints of the edges of the tetrahedron have now disappeared from the picture; Q appears as a triangle with angles $2\pi/3, \pi/3, \pi/3$. However, the fixed points of rotations of order 2 are still the midpoints of the edges, projected on the sphere. The generating rotations u, v of $T_\tau(2,3,3)$ may be presented as unitary matrices U, V defined by

$$U = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{bmatrix}, \quad V = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(-1-i) \\ \frac{1}{2}(1-i) & \frac{1}{2}(1+i) \end{bmatrix}, \quad UV = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}. \tag{2.29}$$

Case 3. The octahedral group $T_\tau(2,3,4)$ and the group $G_\tau(2,3,4)$. The spherical triangle τ with angles $\pi/4, \pi/3, \pi/2$ produces, under repeated reflections in its sides, a tessellation of the sphere by 48 congruent replicas of τ. The reflections r_1, r_2, r_3 in the sides of τ generate a group $G_\tau(2,3,4)$ with the defining relations

$$r_1^2 = r_2^2 = r_3^2 = 1 \tag{2.30}$$

$$(r_1 r_2)^4 = (r_2 r_3)^3 = (r_3 r_1)^2 = 1. \tag{2.31}$$

It has a subgroup $T_\tau(2,3,4)$ generated by elements $u = r_1 r_2$ and $v = r_2 r_3$ with defining relations

$$4u^4 = v^3 = (uv)^2 = 1.$$

u and v are rotations of the sphere which correspond, respectively, to Möbius transformations with matrices U, V defined by

$$U = \begin{bmatrix} (1-i)/\sqrt{2} & 0 \\ 0 & (1+i)/\sqrt{2} \end{bmatrix}, \quad V = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{bmatrix}. \tag{2.32}$$

The matrices U, V themselves generate a nonsplitting central extension of order 48 of the group $T_\tau(4,3,2)$ which can be defined by the relations

$$U^4 = V^3 = (UV)^2. \tag{2.33}$$

$T_\tau(4,3,2)$ is isomorphic with $\sum_4$; it is called the octahedral group because the rotations u, v generate the group of all rotations which carry a regular octahedron, inscribed in the unit sphere, into itself. The group $G_\tau(4,3,2)$ is the direct product of a group of order 2 generated by $(r_1 r_2 r_3)^3$, which represents the central inversion (i.e., which maps every point of the sphere into its diametrically opposite). This was proved by Coxeter (see page 91 of [41]). We shall not reproduce this proof here; but we shall prove the corresponding result in the case of $G_\tau(2,3,5)$. Figure 2.4 shows the tessellation of the sphere by triangles with angles $\pi/4, \pi/3, \pi/2$ and Figure 2.3 shows the tessellation of $\hat{C}$ arising from stereographic projection of the spherical tessellation.

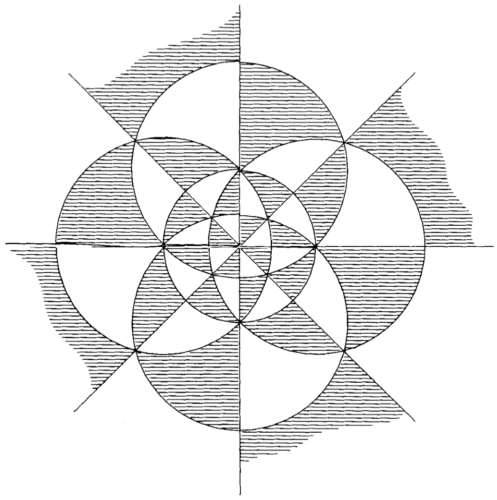

Figure 2.3: Tessellation of the octahedral group on plane.

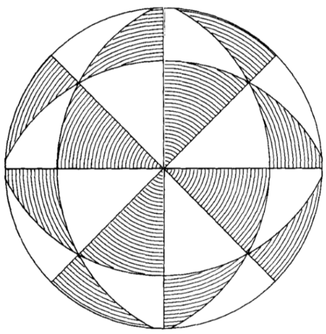

Figure 2.4: Tessellation of the octahedral group on sphere.

Case 4. The icosahedral group $T_\tau(2,3,5)$ and the group $G_\tau(2,3,5)$. This is the most complex and also the most important case of a spherical triangle group and we shall treat it with greater attention to detail than was given to the previous cases. However, we shall still assume without proof the existence of a regular icosahedron which can be inscribed in a sphere. It is true that nobody has doubted the correctness of this assumption since the time of Plato, and that Euclid, in Book 13 of his "Elements" gave a complete construction.

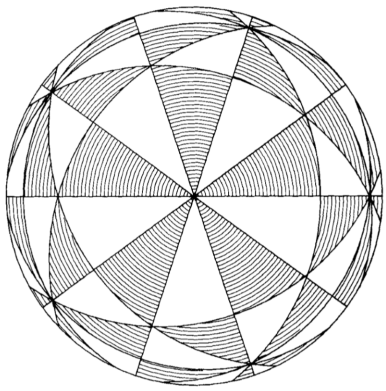

Figure 2.5: Tessellation of the group of the icosahedron on sphere.

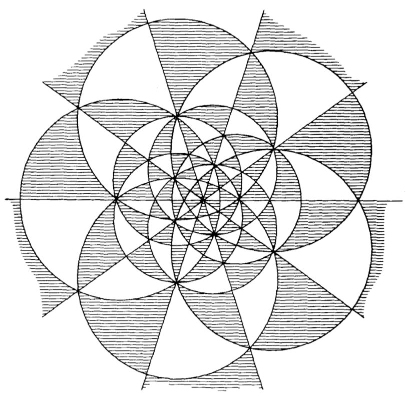

Figure 2.6: Tessellation of the group of the icosahedron on plane.

Figure 2.5 and Figure 2.6 illustrate the tessellation of the sphere with congruent triangles whose angles are $\pi/5, \pi/3, \pi/2$ and the corresponding tessellation of $\hat{C}$ arising from stereographie projection. Their intricate pattern will help us to realize that something has to be proved here. There is no difficulty in proving that we can make a corner out of five equilateral triangles and that, at every one of the five free vertices of this corner, we can again construct a corner congruent to the first one and involving one of its triangles. But it is not obvious that we will obtain a closed polyhedron after we have used up to 20 triangles and that this polyhedron can be inscribed in a sphere. Once the existence of a regular icosahedron has been established, it is not difficult to prove that the group $G_\tau(2,3,5)$ generated by reflections r_1, r_2, r_3 in the sides of a spherical triangle with angles $\pi/5, \pi/2, \pi/3$ contains a subgroup $T_\tau(2,3,5)$ of index 2 consisting of (orientation-preserving) rotations of the sphere isomorphic with the group of rotations which carry the icosahedron into itself. It also follows from elementary geometric arguments that $T_\tau(2,3,5)$ is of order 60, isomorphic with A_5 (the alternating group on five symbols) , and generated by rotations u, v such that

$$u^5 = v^3 = (uv)^2 = 1 \tag{2.34}$$

where

$$u = r_1 r_2, \quad v = r_2 r_3, \quad uv = r_3 r_1. \tag{2.35}$$

The group theoretical results may be summarized as

Theorem 2.3.5. *Let U, V be the 2×2 matrices*

$$U = \begin{bmatrix} -\epsilon^2 & 0 \\ 0 & -\epsilon^2 \end{bmatrix}, \quad V = \frac{-1}{\sqrt{5}} \begin{bmatrix} \epsilon^3 - \epsilon & 1 - \epsilon^4 \\ \epsilon - 1 & \epsilon^2 - \epsilon^4 \end{bmatrix}, \quad (\epsilon = e^{2\pi i/5}). \tag{2.36}$$

Then, U, V define, respectively, fractional linear substitutions u, v which may be interpreted as selfmappings of $\hat{C}$ or as rotations of the unit sphere. The group generated by u, v is a faithful representation of $T_\tau(2,3,5)$. It is defined by the relations (2.34) (that is, by the local relations). The group generated by the matrices U, V itself is a nonsplitting central extension of $T_\tau(5,3,2)$ which will be denoted by G_{120}. Its center is of order 2 and generated by $U^5 = V^3$. It is defined by the relations

$$U^5 = V^3 = (UV)^2. \tag{2.37}$$

The group $G_\tau(2,3,5)$ is defined by the generators r_1, r_2, r_3 and the relations

$$r_1^2 = r_2^2 = r_3^2 = 1, \quad (r_1 r_2)^5 = (r_2 r_3)^3 = (r_3 r_1)^2 = 1. \tag{2.38}$$

It is the direct product of its subgroup $T_\tau(5,3,2)$ with a cyclic group of order 2 generated by

$$C = (r_1 r_2 r_3)^5. \tag{2.39}$$

Proof. The matrices U^5, V^3, and $(UV)^2$ must be $-I$, where I denotes the unit matrix. For U^5, this is obvious; for V^3 and $(UV)^2$, it follows from the fact that the traces of V and UV are, respectively, $2cos(\pi/3)$ and $2cos(\pi/2)$. It is also obvious that U and V are unitary matrices and therefore represent rotations of the sphere. $z = 0$ is one vertex of the icosahedron (after its projection onto $\hat{C}$), and one of the fixed points of UV (both of which lie on the real axis) is the projection of the midpoint of the side of a triangle of the icosahedron whose opposite vertex has been projected onto $z = 0$. It then follows that the homographic substitutions u and v are indeed generators of $T_\tau(2,3,5)$ and satisfy the relations (2.34). It also follows (without geometric considerations) that U, V satisfy (2.37) and that $U^5 = V^3$ generate a center element of order 2 in G_{120}. Furthermore, the group generated by u and v cannot be of an order < 60, since the relations (2.34) show that the group is its own commutator subgroup and therefore not solvable, and the group is not of order 1 since u, v are not the identical selfmappings of $\hat{C}$. From this, it also follows that G_{120} is indeed a group of order at least 120 and, in any case, of twice the order of the group generated by u and v. We can derive our statements about $T_\tau(5,3,2)$ and G_{120} from

Lemma 2.3.6. *The abstract group G_0 on two generators U, V with defining relations (2.37) has a center of order* 2. *Its quotient group in G_0 is isomorphic with A_5 (and G_0 is therefore of order* 120 *and, in fact, the group G_{120}).*

Proof. Let $\omega = (0,1,2,3,4)$ and $\phi = (0,3,1)$ be two permutations on five symbols $0, ..., 4$, written in the usual form as cycles. Then ω, ϕ, and $\omega\phi = (1\ \ 2)(3\ \ 4)$ satisfy

$$\omega^5 = \phi^3 = (\omega\phi)^2 = 1.$$

Since $\omega, \phi \in A_5$, and since the group generated by ω, ϕ cannot be solvable, ω, ϕ generate A_5. Also, the mapping $\mu : U \to \omega, V \to \phi$ defines a homomorphism of G_0 onto A_5, and therefore G_0 has a subgroup H_5 of index 5 consisting of those elements of G_0 that are mapped by μ onto a permutation that fixes the symbol 0. The group of all such permutations in A, is the subgroup A_4 of order 12; it has the elements ω^ν $(\nu = 0,1,2,3,4)$ as right coset repre- sentatives in A_5. Therefore, H_5 has right coset representatives U^ν in G_0, and an application of the Reidemeister-Schreier method gives the generators $R, S_0, ..., S_4$ defined by

$$U^5 = R, \quad VU^{-3} = S_0, \quad UV = S_1, \quad U^2VU^{-2} = S_2,$$

$$U^3VU^{-1} = S_3, \quad U^4VU^{-4} = S_4$$

for H_5. Rewriting the relations

$$U^\nu U^5 V^{-3} U^{-\nu} = 1, \quad U^\nu U^5 (UV)^{-2} U^{-\nu} = 1, \quad (\nu = 0,1,2,3,4)$$

in terms of the generators of H_5, we find as defining relations for H_5

$$\begin{aligned} R = S_0S_3S_1 = S_1S_0S_3 = {S_2}^3 = S_3S_1S_0 = {S_4}^3 = {S_1}^2 = S_2S_3 \\ = S_3S_2 = S_4RS_0 = RS_4S_0. \end{aligned}$$

We can use these relations to express all generators in terms of S_2 and S_4, and we find defining relations for H_5 in terms of these generators to be

$$ {S_2}^3 = {S_4}^3 = (S_2S_4)^2. \tag{2.40}$$

The images of S_2 and S_4 under μ are respectively the permutations $\sigma_2 = (143)$ and $\sigma_4 = (142)$ which generate A_4. In turn, A_4 contains a subgroup V_4 of index three which is the four-group. The preimage of V_4 in H_5 is a normal subgroup H_{15}, of H_5. It consists of those words in S_2, S_4 for which the difference of the sum of exponents of S_4 and of S_2 is divisible by 3. The elements ${S_2}^\lambda$ $(\lambda = 0, 1, 2)$ are right coset representatives of H_{15} in H_5. Using the Reidemeister-Schreier method again, we find that H_{15} is generated by the elements

$$\theta = {S_2}^3, \quad T_0 = S_4S_2, \quad T_1 = S_2S_4, \quad T_2 = {S_2}^2S_4{S_2}^{-1} \tag{2.41}$$

and has as defining relations

$$\theta = T_0\theta^{-1}T_2T_1 = T_1T_0\theta^{-1}T_2 = T_2T_1T_0\theta^{-1} = {T_1}^2 = {T_2}^2 = \theta {T_0}^2\theta^{-1}. \tag{2.42}$$

Using (2.42) to eliminate θ and T_1, we find that H_{15} is generated by T_0, T_2 with defining relations

$${T_0}^2 = {T_2}^2 = (T_0{T_2}^{-1})^2. \tag{2.43}$$

It is easily seen that this is the quaternion group with ${T_0}^2$ as generator of the center and ${T_0}^4 = 1$. Since

$${T_0}^4 = \theta^2 = {S_2}^6 = U^2V^6U^{-2} = 1,$$

it follows indeed that $V^6 = 1$ and that $V^3 = U^5$ generates a center of order 2 in G_0. The quotient group G_0 with respect to its center is A_5 (according to our construction). The center of $G_0 = G_{120}$ cannot be of order > 2, since A_5 has no center. G_{120}, as central extension of A_5, cannot split since otherwise the quaternion group would be a splitting extension of its center, which is not true. This proves 2.3.6 in all details. To complete the proof of 2.3.5, we need

Lemma 2.3.7. *The group G_τ with defining relations (2.38) has a center of order 2 which is generated by the element C in (2.39).*

Proof. The group G_τ contains the group T_τ as subgroup of index two, and T_τ is generated by elements u, v with defining relations (2.34). We know from the proof of 2.3.6 that the map $\mu : u \to (0,1,2,3,4), v \to (0,3,1)$ is an isomorphism. Now we find, using $u = r_1 r_2$ and $v = r_2 r_3$ ($r_1^2 = r_2^2 = r_3^2 = 1$),

$$Cr_3C^{-1}r_3^{-1} = (uv^{-1}u^{-1}v)^2(uv^{-1}uv)(u^{-1}v^{-1}uv)^2, \tag{2.44}$$

$$Cr_1C^{-1}r_1^{-1} = (uv^{-1}u^{-1}v)^2(uv^{-1}u^{-1}v^{-1})(uvu^{-1}v^{-1})^2. \tag{2.45}$$

Since μ maps the elements on the right-hand sides of (2.44) and (2.45) onto the identical permutation, we have shown that C commutes with both r_3 and r_1 and, since it commutes also with $r_1r_2r_3$, must belong to the center of G_τ. We still have to show that $C^2 = 1$. Now

$$(r_1r_2r_3)^{10} = (uv^{-1}u^{-1}v)^5 \to (0,4,1,2,3)^5,$$

where the arrow indicates the action of μ. Therefore $C^2 = 1$. Since G_τ does contain A_5, as a subgroup and since A_5 has no center, $G_\tau(5,3,2)$ is the direct product of A_5 and its center. It is also the triangle group since it is of order 120. This completes the proof of 2.3.5. □

□

□

2.3.3 Tessellations of the Hyperbolic plane

We shall consider triangles with angles $\pi/n_1, \pi/n_2, \pi/n_3$, where n_1, n_2, n_3 are positive integers. A triangle is considered as a closed set (we exclude, for the time being, triangles with an angle equal to zero). Two triangles will be said not to overlap if they have in common no point that is in the interior of either one. "Line" will always mean "hyperbolic straight line" even if we use a model in which the lines are euclidean circles. We summarize our general results as

Theorem 2.3.8. *Let r_1, r_2, r_3 be the reflections in the sides of a hyperbolic triangle τ, with angles $\pi/n_1, \pi/n_2, \pi/n_3$. The images of τ_0, under the action of the distinct elements of the group G_τ generated by r_1, r_2, r_3 fill the hyperbolic plane without gaps and overlappings. G_τ is defined by the local relations*

$$r_1^2 = r_2^2 = r_3^2 = 1, \quad (r_1r_2)^{n_3} = (r_2r_3)^{r_1} = (r_3r_1)^{r_2} = 1. \tag{2.46}$$

Proof. The main difficulty in proving 2.3.8 consists in the proof of the statement that the images of τ_0, fill the hyperbolic plane without gaps and overlappings. For the proof of this part see (pages 177-184 of [42], pages 219-230 of [43] and sections 2, 3, 4, 9, Chapter 3 of [44]). We will prove only the group theoretical part of the theorem.

Let us assume first that the integers n_1, n_2, n_3 are different from each other. In this case, the following is true: Let $\Theta = W(r_1, r_2, r_3)$ be a word in the generators r_1, r_2, r_3 of G_τ defining a particular selfmapping (noneuclidean motion) of the hyperbolic plane. Let $\tau = \Theta(\tau_0)$ be the image of τ_0 under the action of Θ. Since the images of τ_0, under the action of the elements of G_τ fill the plane without gaps and overlappings, τ and τ_0, can have an interior point in common only if τ covers τ_0. But in this case Θ must be the identical selfmapping since the vertices of τ_0 must be mapped onto themselves (otherwise angles of different size would have to be congruent to each other), and a mapping which fixes three points is the identity if the points are not collinear.

From here to the end of the proof of 2.3.9, Figure 2.9 provides the basis for illustrations. Figure 2.9 shows the tessellation of the hyperbolic plane with triangles where $n_1 = 2, n_2 = 3, n_3 = 7$.

We use 2.3.2 to assign a chain C of triangles τ_ν, $(\nu = 0, ..., n)$ to $\Theta = W(r_1, r_2, r_3)$. Since r_1, r_2, r_3 are of order two, we may assume that W is mitten in the form

$$W = g_1 g_2 ... g_n, \tag{2.47}$$

where each g_ν $\nu = 1, ..., n)$ denotes either r_1, r_2, or r_3 and where g_ν and $g_{\nu+l}$ never denote the same generator. We identify I_0 of 2.3.2 with τ_0, and I_ν, with its image τ_ν, and C consists of a sequence of congruent triangles τ_ν $(\nu = 0, ..., n)$ each of which arises from the previous one by reflection in one of its sides. Since $g_\nu \neq g_{\nu+l}$, it never happens that $\tau_{\nu+1}$ is reflected back into τ_ν. In other words, $\tau_\nu \neq \tau_{\nu+2}$. Of course, $\tau_0 = \tau_n$. With the chain C we associate a closed polygon Π which is constructed as follows: We mark a point P_0 in the interior of τ_0 and denote by P_ν its image in τ_ν. We also mark the midpoints ${Q_\lambda}^{(0)}$ $(\lambda = 1, 2, 3)$ of the sides of τ_0 and we denote the corresponding midpoints of the sides of τ_ν by ${Q_\lambda}^\nu$. We join P_ν, and $P_{\nu+l}$ with the particular midpoint ${Q_\lambda}^\nu$ of the side which τ_ν and $\tau_{\nu+1}$ have in common by segments of straight lines which will be, respectively, contained in τ_ν and in $\tau_{\nu+1}$ because of the convexity of triangles. Then Π is the oriented polygon starting in P_0 and going, via midpoints and points $P_1, ..., P_n$, back to P_0. This polygon Π may intersect itself. If Π should intersect itself, it must happen that a triangle τ_ν coincides with a triangle $\tau_{\nu+1}$, where

$$0 \leq \nu < \nu + \mu \leq n$$

and either $0 < \nu$ or $\nu + \mu < n$ (or both). In this case, a subword of μ symbols g_p in W represents the unit element of G_τ, and if we choose a shortest nontrivial subword of this type we can associate with it a simple (i.e., non self intersecting) polygon Π. We shall assume that the polygon Π associated with our original W already has this property.

Since Π encloses a finite area, only finitely many triangles belonging to the tessellation of the hyperbolic plane can have points in the interior of Π, since each triangle has a positive area and since, at every vertex of a triangle inside Π, only finitely many triangles of the tessellation meet since Π itself passes only through n triangles τ_ν. Of these, every τ_ν has at least one and at most two vertices in the interior I^* of Π, since Π intersects exactly two sides of each τ_ν and each of these two sides must have one endpoint inside and the other one outside of Π.

Let P be a particular vertex inside Π belonging to one of the triangles of C, and let $\tau_\nu, ..., \tau_{\nu+k}$ be the consecutive triangles of the chain forming Π which meet in P. (If $\nu + \mu \geq n$, we reduce $\nu + \mu$ mod n to a number between 0 and $n-1$.) We have $k > 0$ since at least two consecutive triangles of the chain C meet at P, namely those that have a side in common for which P and a point outside of Π are the endpoints. Let $\tau_\nu, \tau_{\nu+1}, ..., \tau_{\nu+k}; {\tau_1}', ..., {\tau_s}'$ be the set of distinct triangles of our triangulation which meet in P, and assume that they follow each other in the order in which they are written down in such a manner that each has a side in common with the previous one and that ${\tau_s}'$ has a side in common with τ_ν. (Of course, we may have $s = 0$.) We shall say that these triangles form the star of P. Now we replace in C the triangles $\tau_\nu, \tau_{\nu+1}, ..., \tau_{\nu+k}$ by the sequence $\tau_\nu, {\tau_s}', ..., {\tau_1}', \tau_{\nu+k}$ and adjust Π accordingly, obtaining thereby a new polygon Π'. Then P is outside of Π', but Π' itself contains no point outside of Π. It is not necessary anymore that Π' should be a simple polygon, but it is again the union of simple polygons connected either in points or by polygons without any interior. These then consist of combinations of line segments through which Π' runs in both directions. Since the total number of vertices of tesselating triangles in the interior of Π' is less than the corresponding number for Π, our process, if continued, must terminate after finitely many steps with a "one-dimensional" polygon Π^* which has no interior at all.

Now we shall interpret these geometric considerations in algebraic terms. The replacement of the triangles $\tau_\nu, \tau_{\nu+1}, ..., \tau_{\nu+k}$ by triangles $\tau_\nu, {\tau_s}', ..., {\tau_1}'$, $\tau_{\nu+k}$ means that somewhere in the word $W(r_1, r_2, r_3)$ which defines Π we have replaced k symbols of one of the cyclically written words

$$(r_1 r_2)^{n_3}, \quad (r_2 r_3)^{n_1}, \quad (r_3 r_1)^{n_2}$$

by the inverse of the product of the remaining symbols-which we are allowed to do group theoretically in view of the defining relations for G_τ given in 2.3.8. (Note that not the triangles but the transitions from one triangle to the next are defined by the generators r_1, r_2, r_3.) Doing this repeatedly will eventually change our word W into a word W^* which defines a polygon Π^* without any interior. But then Π^* must be such that, at one point, it consists of a segment immediately followed by the same segment with the opposite orientation. This means that one of the generators r_1, r_2, r_3 in W^* is followed by the same generator. In view of the defining relations $r_1^2 = r_2^2 = r_3^2 = 1$, these two symbols in W^* can be omitted, and a continuation of the same process must lead to the empty word.

We have thus proved 2.3.8 in the case where n_1, n_2, n_3 are distinct integers. Suppose now that we have $n_2 = n_3$. Then we can bisect τ_0 by drawing one of its altitudes, obtaining a triangle τ_0' (with angles $\pi/2n_1, \pi/n_2, \pi/2$. These angles are distinct unless $n_2 = 2n_1$, since $n_2 > 2$ if $n_2 = n_3$. In the case where $n_2 = 2n_1$, we repeat the same process, obtaining now a triangle τ_0'' with angles

$$\pi/2n_1, \quad \pi/2n_2, \quad \pi/4$$

which are necessarily distinct because $n_2 > 2, n_2 = 2n_1$, and the case $2n_1 = 4$ cannot arise if $n_2 = 2n_1$ since $n_2 > 2$.

Suppose now that τ_0' has three distinct angles. Then 2.3.8 applies, and τ_0' is the fundamental region of a group T_τ' generated by three reflections r_1, r_3, r_4 with defining relations

$$r_1^2 = r_3^2 = r_4^2 = 1, \quad (r_1r_3)^{n_2} = (r_3r_4)^{2n_1} = (r_4r_1)^2 = 1,$$

where P is the reflection in the altitude of τ_0 which is a side of τ_0'. The original τ_0 consists of τ_0' and its image τ_1' under the action of r_4. We find (from direct geometric considerations) that

$$r_4r_3r_4 = r_2.$$

Now all we have to show is that the elements $r_1, r_3, r_4r_3r_4$ of T_τ' generate a subgroup G_τ of index 2 with coset representatives $1, r_4$ and with the defining relations for G_τ where now $r_2 = r_4r_3r_4$. The Reidemeister-Schreier method shows that G_τ is indeed generated by $r_1, r_3, r_4r_1r_4$, and $r_4r_3r_4 = r_2$. Because of $(r_4r_1)^2 = 1, r_4r_1r_4 = r_1$ is redundant. The defining relations are

$$r_1^2 = r_3^2 = 1 = (r_4r_1r_4^{-1})^2 = (r_4r_3r_4^{-1})^2 = r_2^2 = 1$$

and

$$(r_1r_3)^{n_2} = 1, \quad (r_4r_1r_3r_4^{-l})^{n_2} = (r_4r_1r_4^{-1}r_4r_3r_4^{-1})^{n_3} = (r_1r_2)^{n_2} = (r_1r_2)^{n_3} = 1,$$

$$(r_3r_4)^{2n_1} = (r_3r_4r_3r_4)^{n_1} = (r_2r_3)^{n_1} = 1, \quad (r_4r_3)^{2n_1} = (r_4r_3r_4r_3)^{n_1} = (r_2r_3)^{n_1} = 1.$$

These relations either agree with those for G_τ or are derivable from them. This settles the case where τ' has three distinct angles. In the case where only τ'' has three distinct angles, we have to build up τ_0 from four replicas of τ_0'' and we will obtain G_τ as a subgroup of index four of the group of reflections in the sides of τ_0''. This then completes the proof of 2.3.8. □

We can use the geometric considerations in the proof of 2.3.8 to solve the word problem for the group G_τ in a simple algebraic manner. We shall prove:

Theorem 2.3.9. *Let n_1, n_2, n_3 be positive integers such that*

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} < l.$$

Let W be a word in the generators r_1, r_2, r_3 of the group G_τ defined in 2.3.8 and assume that W is normalized in the form (2.47). Then, if $W = 1$, there appears somewhere in W a subword W' of one of the cyclically written relators

$$(r_1r_2)^{n_3}, \quad (r_2r_3)^{n_1}, \quad (r_3r_1)^{n_2} \tag{2.48}$$

whose length exceeds half of the length of the relator.

Proof. 2.3.9 allows a solution of the word problem for G_τ since it shows that we can shorten every word $W = 1$ by replacing the larger part of a relator (which appears as a subword in W) with the inverse of the shorter part of the same relator. The resulting word W' can then be put into the normalized form (2.47) without increasing its length, by using the relations $r_1^2 = r_2^2 = r_3^2 = 1$. If $W = 1$, a repetition of this process must change W into the empty word. To prove 2.3.9, we return to the closed Π associated with W and defined in the proof of 2.3.8. We inscribe within Π an interior polygon $I\Pi$ the sides of which are also sides of the triangles τ_ν and which is defined as follows.
Let Q_1 be the vertex of τ_1 inside Π which is also a vertex of τ_2. Assume that Q_1 is also a vertex of $\tau_3, ..., \tau_{k_1}$ but not a vertex of τ_{k_1+1}. Then τ_{k_1+1} must have a second vertex Q_2 inside Π and Q_2 must also be a vertex of τ_{k_1+2}. We continue in this manner, obtaining an oriented closed poygon $I\Pi$ with vertices $Q_1, Q_2, ..., Q_h$ which is entirely inside Π. If $I\Pi$ has only one vertex, then W must be a cyclic permutation of one of the relators (2.48) , and our theorem is proved. Otherwise, $I\Pi$ must enclose some area since the normalization of W excludes the case that an interior angle of $I\Pi$ can be zero. (Of course, $I\Pi$ need not be a simple polygon and may enclose several finite parts of the plane, all of which are fully tesselated with replicas of τ_0.)

Assume now that the oriented part $I\Pi^*$ of $I\Pi$ qith vertices $Q_s, Q_{s+1}, ..., Q_{s+t}$ is a simple closed polygon which encloses some area. Then at least one of the interior angles of this $(t+1)$-gon must be less than π since, generally speaking, the sum of the interior angles of a hyperbolic polygon with $(t+1)$ vertices is less than $(t-1)\pi$. Let Q be a point where the interior angle a is less than π. Then α will be a multiple of π/N, where N denotes one of the numbers n_1, n_2, n_3, and there will be $2[1-(\alpha/\pi)]N > N$ consecutive triangles τ_ν which belong to the chain C, and which meet in Q. The corresponding subword W' of W satisfies the conditions of 2.3.9. □

An important result about the structure of the triangle group is

Theorem 2.3.10. *Let T_τ be the subgroup of index 2 in the triangle group G_τ which consists of orientation-preserving selfmappings of the noneuclidean plane; T_τ is generated by $A = r_1 r_2$ and $B = r_2 r_3$ and defined by the relations*

$$A^{n_3} = B^{n_1} = (AB)^{n_2} = 1. \tag{2.49}$$

Then the elements $\neq 1$ of finite order in T_τ are conjugates of powers of A, B, or AB.

Proof. The elements of T_τ map onto itself the tessellation of the noneuclidean plane defined by the fundamental region FR of T_τ. Suppose $W \neq 1$ is of finite order. It has exactly one fixed point ϕ in the noneuclidean plane. Let FR^* be an image of FR such that ϕ lies inside or on the boundary of FR^*, and let Θ be the element of T_τ which maps FR onto FR^*. Then $W^* = \Theta W \Theta^{-1}$ has a fixed point ϕ^* inside or on the boundary of FR. If ϕ^* were an inner point of FR, then there would exist distinct points (close to ϕ^*) inside FR such that W^* would map one of them onto another one. This contradicts the definition of FR. Therefore, ϕ^* is on the boundary of FR and W^* must map FR onto one of its images that have a point in common with FR. But then ϕ^* must be one of the vertices of FR. These, in turn, are the fixed points of the powers respectively of A, B, AB, and BA. □

Corollary 2.3.11. *If N is a normal subgroup of T_τ such that, under the homomorphism $T_\tau \to T_\tau/N$, the elements A, B, AB of T_τ are respectively mapped onto elements of the same order n_1, n_2, n_3 in T_τ/N, then N is torsion free (i.e., the unit element is the only element of finite order).*

Proof. The conjugates of powers $\neq 1$ of A, B, AB do not belong to N but to other cosets of N in T_τ since their images under the mapping $T_\tau \to T_\tau/N$ are not the unit element of T_τ/N.

Although it can be shown in general that a noneuclidean triangle with angles α, β, γ exists and is uniquely determined if α, β, γ are nonnegative and $\alpha, \beta, \gamma < \pi$, we shall give an explicit construction of a right triangle and an explicit representation of the subgroup of proper noneuclidean motions in the group generated by reflections in the sides of the triangle.

We shall use the unit disk as a model for noneuclidean geometry, with the circles orthogonal to the unit circle serving as straight noneuclidean lines. The angles will then be the euclidean angles. We summarize our results as:

Theorem 2.3.12. *Let $\alpha > 0$ and $\beta \geq 0$ be angles such that $\alpha + \beta < \pi/2$. Let O, Q, P be three points in the unit disk $|z| < 1$, defined respectively by their coordiantes*

$$z_O = 0, \quad z_Q = x_Q, \quad z_P = x_P + iy_P,$$

where

$$x_Q = (\cos\beta - \sin\alpha)/\rho, \quad x_P = [\cos\alpha\cos(\alpha+\beta)]/\rho,$$
$$y_P = [\sin\alpha\cos(\alpha+\beta)]/\rho$$

and

$$\rho = (\cos^2\beta - \sin^2\alpha)^{1/2}.$$

Then O, Q, P are the vertices of a noneuclidean triangle with angles $\alpha, \pi/2, \beta$, respectively, at O, Q, P. The sides OQ and OP are respectively parts of the real axis and of the straight euclidean line joining O and P. The side QP is part of the circle with center at $z = x_c$, where $x_c = (\cos\beta)/\rho$, and with radius $r = (\sin\alpha)/\rho$.

Let r_1, r_2, r_3 denote, respectively, the reflections in the sides OQ, QP, PO of the triangle. They generate a group G_τ which has a subgroup T_τ of index 2 consisting of orientation-preserving noneuclidean motions and is generated by

$$a = r_1 r_2, \quad b = r_2 r_3. \tag{2.50}$$

To a, b there correspond respectively matrices A, B of Mobius transformations which are given by

$$A = \frac{i}{\sin\alpha}\begin{bmatrix} \cos\beta & \rho \\ -\rho & -\cos\beta \end{bmatrix}, \quad B = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} \tag{2.51}$$

and which map $|z| = 1$ onto itself. If $\alpha = \pi/n_2$ and $\beta = \pi/n_1$, where n_1, n_2 are integers, then A, B define a group T_τ of Mobius transformations with the defining relations

$$A^2 = B^{n_2} = (AB)^{n_1} = 1. \tag{2.52}$$

Proof. See page 87 of [38]. $\square$

$\square$

Corollary 2.3.13. *The following groups on two generators A, B withdefining relations*

$$\begin{array}{rl} (i) & A^2 = B^6 = (AB)^4 = 1 \\ (ii) & A^2 = B^6 = (AB)^6 = 1 \\ (iii) & A^2 = B^8 = (AB)^8 = 1 \end{array}$$

have faithful representations in terms of Mobius transformations which map $|z| \leq 1$ onto itself. They are represented respectively by the following matrices:

$$(i)\ A = i\begin{bmatrix} \sqrt{2} & -1 \\ 1 & -\sqrt{2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2}(\sqrt{3}+i) & 0 \\ 0 & \frac{1}{2}(\sqrt{3}-i) \end{bmatrix}$$

$$(ii)\ A = i\begin{bmatrix} \sqrt{3} & \sqrt{2} \\ \sqrt{2} & -\sqrt{3} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2}(\sqrt{3}+i) & 0 \\ 0 & \frac{1}{2}(\sqrt{3}-i) \end{bmatrix}$$

$$(iii)\ A = i\begin{bmatrix} 1+\sqrt{2} & [2(\sqrt{2}+1)]^{1/2} \\ -[2(\sqrt{2}+1)]^{1/2} & -1-\sqrt{2} \end{bmatrix},$$

$$B = \frac{1}{2}\begin{bmatrix} (2+\sqrt{2})^{1/2} + i(2-\sqrt{2})^{1/2} & 0 \\ 0 & (2+\sqrt{2})^{1/2} - i(2-\sqrt{2})^{1/2} \end{bmatrix}.$$

Proof. See page 88 of [38]. $\square$

Note that in all cases the entries are algebraic integers, but that in case (iii) they do not all belong to a cyclotomic field since

$$[2(\sqrt{2}+1)]^{1/2}/(2+\sqrt{2})^{1/2} = 2^{1/4},$$

and the splitting field of $x^4 - 2 = 0$ has a nonabelian Galois group over the rationals. We can replace A, B by real matrices, but then we lose their property of having algebraic integers as entries. The entries in A and B will be algebraic integers for rational values of α/π and β/π if and only if $g = (\cos\beta)/(\sin\alpha)$ is an algebraic integer. If $\alpha = \pi/n_2$ and $\beta = \pi/n_1$, and if M is the least common multiple of n_1, n_2, we have

$$g = i\frac{\epsilon^{2\lambda}+1}{\epsilon^{2\mu}-1}\epsilon^{\mu-\lambda}, \quad (\epsilon = e^{\pi i/M}),$$

where $\mu n_2 = \lambda n_1 = M$. According to a lemma in the theory of cyclotomic fields, the following is true: Let $\zeta = e^{\pi i/N}$. Then $1 - \zeta^\nu$ will be a unit if ν and N are coprime and N is divisible by at least two distinct prime numbers. If, however, N is a power of a prime number, then $(\zeta^\nu - l)/(\zeta - 1)$ will be a unit and, therefore, an algebraic integer. For a proof, see (page 203 of [45]). This remark allows us to show in many cases that the matrices in (2.51) have entries which are algebraic integers.

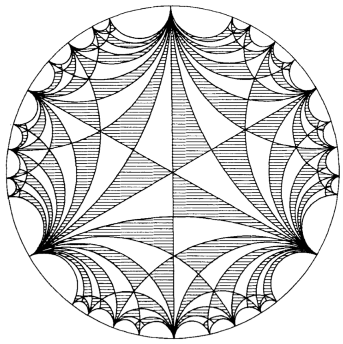

Figure 2.7: Tessellation of the disk by triangles of angles $\pi/2, \pi/3, 0$.

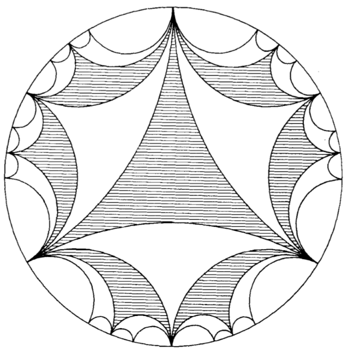

Figure 2.8: Tessellation of the disk induced by repeated reflection of zero-angle triangle.

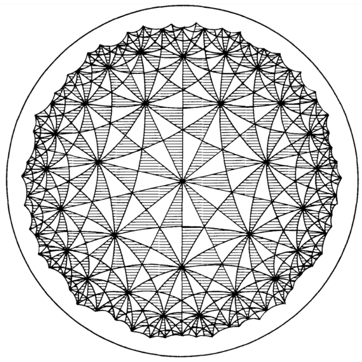

Figure 2.9: Tessellation of the disk by triangles of angles $\pi/2, \pi/3, \pi/7$.

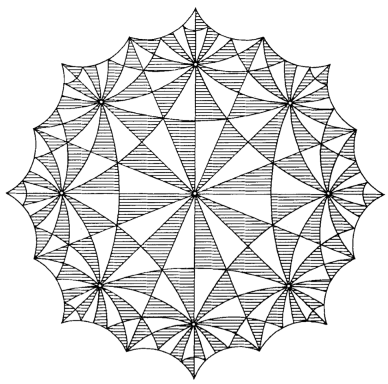

Figure 2.10: Tessellation of the disk by triangles of angles $\pi/2, \pi/3, \pi/8$.

Chapter 3

On the Generic Triangle Group

Let us now summarize the discussion. We have defined the metric as a product of matrices; a point is given from the scalar product of two vectors. Let us recall the definition of Euclidean metric given in (1.1):

$$\bar{g} = \sum_i dx^i dx^i = \sum_i (dx^i)^2 = \sum_{i,j} \delta_{ij} dx^i dx^j.$$

We have matrices who are initially the same. Let us now add a factor k such that we have then the family of metrics given by:

$$g(v,v) = \begin{cases} \frac{4r^4|v|^2}{(r^2-|x_2|^2)^2} = \frac{4r^4}{(r^2-|x_2|^2)^2}\delta_{ij}dv^i dv^j & \text{for } k<0 \\ \delta_{ij}dv^i dv^j & \text{for } k=0 \\ \frac{4r^4|v|^2}{(r^2+|x_2|^2)^2} = \frac{4r^4}{(r^2+|x_2|^2)^2}\delta_{ij}dv^i dv^j & \text{for } k>0 \end{cases},$$

where for $k < 0$ we have the hyperbolic metric of the disk of Poincaré of radius r given in 1.2.10, for $k = 0$ we have the Euclidean metric and for $k > 0$ we have the pullback metric of the round metric through the stereographic projection of the sphere given by (1.15).
We know from 1.2.3 that the geodesics of the spherical space are the great circles, intersections of S^n_r with 2-planes through the origin, with constant speed parametrizations and from 1.2.12 that the geodesics of the hyperbolic space are the line segments through the origin and the circular arcs that intersect $\partial\mathbb{D}^n_r$ orthogonally.
We also know from (1.16) and (1.19)that the curvature of the spherical space is $\frac{1}{r^2}$ and the curvature of the hyperbolic space is $-\frac{1}{r^2}$.
Let us now increase the radius r of the hyperbolic and spherical space, $r \to \infty$ and compute the limits:

$$\lim_{r\to\infty} \frac{1}{r^2} = \lim_{r\to\infty} -\frac{1}{r^2} = 0.$$

Thus, if we increase the radius r, both the spherical and hyperbolic space tend to have curvature $k = 0$ and become flat like the Euclidean space.
By continuously varying k we have that also the points and the matrices continuously vary.
Let us now fix three points and construct the geodesics which join these points. The geodesics continuously vary according to the matrices. Thus, there is not continuity only in matrices but also in the geodesics which construct the triangles. Therefore, for $k \to 0$, i.e., for $r \to \infty$ the geodesics of the spherical and hyperbolic space tend to the geodesics of the Euclidean space, i.e., they tend to become straight lines.
As we have seen in chapter 2 the groups of triangles are generated by the reflections of the sides of triangles. Now, varying the metric we have that also the triangle varies, which implies that also the reflections continuously vary. This by itself implies that if we consider a relation, i.e., product of generators equal to identity, the result of this product continuously varies because it is a composition of reflections who continuously vary.
Now, if a relation is true in all the hyperbolic and spherical spaces, it is true also in the Euclidean space. Thus, from the fact that the Euclidean space is stable in the sense that we will define in sequel, we have that also the hyperbolic and spherical space are stable.

3.1 Generic Triangles

Given an Euclidean triangle $\tau = A_1A_2A_3$, let $l_i > 0$ denote the length of the edge $e_i = A_jA_k$ and $\alpha_i > 0$ denote the misure in radians of the (non-oriented) interior angle $A_jA_iA_k$, with $\{i, j, k\} = 1, 2, 3$.

Definition 3.1.1. We call τ a *generic* triangle if for some $k > 0$ (hence for almost every $k \in \mathbb{R}$) the real numbers kl_1, kl_2 and kl_3 are algebraically independent (over the rationals), namely it does not exist any non-trivial polynomial $p(x_1, x_2, x_3) \in \mathbb{Z}[x_1, x_2, x_3]$ such that $p(kl_1, kl_2, kl_3) = 0$.

A different formulation of the above condition is that for every non-trivial polynomial $p(x_1, x_2, x_3) \in \mathbb{Z}[x_1, x_2, x_3]$ the polynomial $q(x) = p(l_1x, l_2x, l_3x)$ is non-trivial in $\mathbb{R}[x]$, or equivalently the field $\mathbb{Q}(l_1x, l_2x, l_3x)$ has transcendence degree 3 over $\mathbb{Q}$ (see [[6], Sec. 6.4]).

Proposition 3.1.2. *The set of generic triangles is a dense G_δ-subset in the space of all Euclidean triangles.*

Proof. See page 2 of [1]. □

Proposition 3.1.3. *An Euclidean triangle τ as above is a generic triangle if and only if one of the following equivalent properties holds:*
(s) for some (hence almost every) $k \in \mathbb{R}$ the real numbers $k \sin \alpha_1, k \sin \alpha_2$, and $k \sin \alpha_3$ are algebraically independent over the rationals;
(c) for some (hence almost every) $k \in \mathbb{R}$ the real numbers $k \cos \alpha_1, k \cos \alpha_2$, and $k \cos \alpha_3$ are algebraically independent over the rationals.

Proof. See page 2 of [1]. □

Lemma 3.1.4. *If an algebraic relation $p(\sin \alpha_1, \sin \alpha_2, \sin \alpha_3, \cos \alpha_1, \cos \alpha_2, \cos \alpha_3) = 0$, with $p(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}[x_1, x_2, x_3, x_4, x_5, x_6]$, holds for a generic triangle, then it holds for every triangle.*

Proof. See page 3 of [1]. □

A deeper analysis of the algebraic dependence of $\cos \alpha_1, \cos \alpha_2$ and $\cos \alpha_3$, shows that generic triangles are typical, in the sense of the following definition (see [[2], Sec. 6.B]).

Definition 3.1.5. An Euclidean triangle τ as above is called a *typical* triangle if the real numbers α_1, α_2 and α_3 are linearly independent over the rationals.

Proposition 3.1.6. *Generic Euclidean triangles are typical.*

Proof. See page 3 of [1]. □

3.2 The Triangle Group

For any Euclidean triangle τ we denote by $G_\tau = \langle r_1, r_2, r_3 \rangle$ the subgroup of the group E(2) of the Euclidean isometries of the plane generated by the refections r_1, r_2, and r_3 across the edges e_1, e_2, and e_3 of τ, respectively. We call G_τ the *triangle group* of τ.
The standard exact sequence

$$1 \to \mathbb{R}^2 \xrightarrow{\imath} E(2) \xrightarrow{\lambda} O(2) \to 1,$$

where $\imath$ is the inclusion of $\mathbb{R}^2$ in $E(2)$ as the subgroup of translations and λ is the linearization homomorphism, induces by restriction the exact sequence

$$1 \to T_\tau \xrightarrow{\imath_\tau} G_\tau \xrightarrow{\lambda_\tau} S_\tau \to 1, \tag{3.1}$$

where $T_\tau \subset G_\tau$ is the translation subgroup consisting of all translations in G_τ, while $S_\tau = \lambda(G_\tau) = \langle s_1, s_2, s_3 \rangle \subset O(2)$ with $s_i = \lambda(r_i)$ the linearization

of r_i.
We observe that the structure of the group G_τ (including the latter exact sequence) is invariant under similarities. Hence, without loss of generality we can assume that the incircle of the triangle τ coincides with the unit circle centered at the origin. Under this assumption, we have

$$(x)r_i = (x)s_i + 2v_i \tag{3.2}$$

for every $x \in \mathbb{R}^2$ (here and in the following we use the right notation for the action of G_τ), where v_i is the unit vector from the origin to the tangency point of the edge e_i and the incircle of τ.
We want to determine a minimal presentation of the group S_τ in the case when τ is a typical, and hence S_τ is a dense subgroup of $O(2)$. In order to do that, we first recall from [[2], Sec. 6.A] the definition of a stable sequence and the stability criterion for a product of generators of S_τ to be trivial.

Definition 3.2.1. A sequence $i_1 i_2 ... i_n$ of symbols from $1, 2, ..., N$ is called a *stable sequence* if its terms can be paired into disjoint pairs of identical symbols, one located at an odd and the other at an even position. Differently said, the length n of the sequence is even and the symbolic alternating sum $i_1 - i_2 + ... + i_{n-1} - i_n$ vanish (as an algebraic sum of symbols, not of integers).

Lemma 3.2.2. *A sequence $i_1 i_2 ... i_n$ is stable if and only if one can reduce it to the empty sequence by a finite number of operation of the following types:*
(a) transposition of two adjacent subsequences both consisting of two symbols;
(b) deletion of a subsequence consisting of two identical symbols.

Proof. First of all, we note that both operations and the inverse of the second one, that is the insertion of two adjacent identical symbols in a sequence, all preserve the parity of the position of each term in the sequence, hence they preserve stability. This immediately gives the "if" part of the statement, since the empty sequence is stable. The "only if" part can be proved by induction on the length of the sequence, starting once again from the empty sequence. For the inductive step, assume we are given any non-empty stable sequence $i_1 i_2 ... i_n$. The stability implies that $i_{2k-1} = i_2$ for some $1 \leq k \leq n/2$. If $k = 1$, we can reduce the length of the sequence by deleting the subsequence $i_1 i_2$. Otherwise, by $k - 2$ transpositions of pairs, we get a sequence starting with the four symbols $i_1 i_2 i_{2k+1} i_{2k+2}$, then we can reduce the length of the word by deleting the subsequence $i_2 i_{2k+1}$. □

Lemma 3.2.3. *If a sequence $i_1 i_2 ... i_n$ of symbols from $\{1, 2, 3\}$ is stable, then the product $s_{i_1} s_{i_2} ... s_{i_n}$ is the identity in S_τ. Moreover, for a typical triangle*

τ the stability of the sequence $i_1i_2...i_n$ is also necessary in order $s_{i_1}s_{i_2}...s_{i_n}$ to be the identity.

Proof. We proceed in the same spirit as in [[2], Sec. 6.B]. We first orient the egdes e_1, e_2 and e_3 in the counterclockwise way along the boundary of the triangle τ, and denote by β_i the oriented angle from e_1 (fixed as reference vector) to e_i. Then, we have $\beta_1 = 0, \beta_2 = \pi - a_3$ and $\beta_3 = \pi + a_2 \mod 2\pi$. Moreover, any composition $s_j s_k$ gives the linear rotation of angle $2(\beta_k - \beta_j) \mod 2\pi$, and hence any product $s_{i_1}s_{i_2}...s_{i_n}$ with n even gives the linear rotation of angle

$$\phi = 2(\beta_{i_1} + \beta_{i_2}) + ... + 2(\beta_{i_n} + \beta_{i_{n-1}}) \mod 2\pi. \tag{3.3}$$

Now, the stability of the sequence $i_1i_2...i_n$ implies that $\phi = 0 \mod 2\pi$ and thus $s_{i_1}s_{i_2}...s_{i_n}$ is the identity in S_τ.
In the opposite direction, start with a product $s_{i_1}s_{i_2}...s_{i_n}$ that gives the identity. Then, n must be even and (3.3)can be rewritten in terms of the α_i's by using the above identities. If the sequence $i_1i_2...i_n$ is not stable, this yields a non-trivial rational linear relation among the angles α_2, α_3 and π, which implies that τ is not typical. □

At this point, we are in position to obtain the wanted presentation of S_τ.

Proposition 3.2.4. *For a typical triangle τ the group S_τ admits the finite presentation $\langle x_1, x_2, x_3 | x_1^2, x_2^2, x_3^2, (x_1x_2x_3)^2 \rangle$, with the symbols x_1, x_2, x_3 corresponding to s_1, s_2, s_3, respectively.*

Proof. According to 3.2.3, all the four relations of the presentation hold in S_τ, because the corresponding sequences of indices are stable. Viceversa, 3.2.2 and 3.2.3 say that any word $x_{i_1}x_{i_2}...x_{i_n}$ representing the identity in S_τ can be reduced to the empty word by canceling squared terms x_i^2 and commuting products x_ix_j and x_kx_l. So, to conclude the proof it is enough to show that any commutator $[x_ix_j, x_kx_l] = (x_ix_j)^{-1}(x_kx_l)^{-1}(x_ix_j)(x_kx_l)$ is the identity modulo the given four relations. Up to inversions, the only non-trivial cases are $[x_1x_2, x_1x_3], [x_1x_2, x_2x_3]$ and $[x_1x_3, x_2x_3]$. For these we have: $[x_1x_2, x_1x_3] = x_3(x_1x_2x_3)^{-2}x_3, [x_1x_2, x_2x_3] = x_2x_1(x_1x_2x_3)^{-2}x_1x_2$ and $[x_1x_3, x_2x_3] = x_3x_1(x_1x_2x_3)^{-2}x_1x_3$. □

3.3 The Translation Subgroup

Due to 3.2.3 and the exact sequence (3.1), for any Euclidean triangle τ the product $r_{i_1}r_{i_2}...r_{i_n}$ gives a translation in T_τ if the sequence $i_1i_2...i_n$ is stable.

On the other hand, when the triangle τ is typical we obtain in this way all the translations in T_τ, and it is clear from 3.2.4 that a special role is played by the minimal stable product $t_1 = (r_1 r_2 r_3)^2$, coming from the code-word of the Fagnano trajectory, the simplest stable periodic trajectory in any acute triangle (see [[2], Sec. 2.A]). Notice that the translation t_1 is non-trivial for any (non-degenerate) triangle τ, as it easily follows by elementary geometry. We denote the conjugation class of t_1 in G_τ by $C(t_1) = \{((t_1))g = g^{-1}t_1 g | g \in G_\tau\} \subset T_\tau$. According to equation (3.2), by identifying translations in T_τ with the corresponding vectors in $\mathbb{R}^2$ and considering the natural action of $S_\tau \subset O(2)$ on them, for $g = r_{i_1} r_{i_2} ... r_{i_n} \in G_\tau$ the conjugate $((t_1))g$ is given by

$$((t_1))r_{i_1} r_{i_2} ... r_{i_n} = ((t_1))s_{i_1} s_{i_2} ... s_{i_n}. \tag{3.4}$$

In the case when τ is a typical triangle, the density of the subgroup $S_\tau \subset O(2)$ implies that $C(t_1)$ forms a dense subset of the circle $\rho S^1 \subset \mathbb{R}^2$ of radius

$$\rho = ||t_1|| = 4(\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3).$$

Theorem 3.3.1. *For a typical triangle τ the translation subgroup $T_\tau \subset G_\tau$ is normally generated by the translation $t_1 = (r_1 r_2 r_3)^2$. Moreover, if τ is generic then T_τ is a free abelian group having as a basis the conjugation class $C(t_1)$.*

Proof. Given any $t \in T_\tau$ with τ a typical triangle, we can express it as a product $r_{i_1} r_{i_2} ... r_{i_n}$ of generators of G_τ. In view of the exact sequence (3.1), the corresponding product $s_{i_1} s_{i_2} ... s_{i_n}$ gives the identity in S_τ, hence the sequence $i_1 i_2 ... i_n$ is stable by 3.2.3. Then, arguing as in the proof of 3.3.1, we can rewrite $r_{i_1} r_{i_2} ... r_{i_n}$ as a product of conjugates of r_1^2, r_2^2, r_3^2 and $(r_1 r_2 r_3)^2$. Since the r_i^2's are trivial in G_τ, we can conclude that t is a product of conjugates of t_1, which gives the first part of the theorem. Now, assume that τ is a generic triangle. We have to show that $C(t_1)$ is linearly independent over $\mathbb{Z}$, that is the only vanishing linear combination of pairwise distinct elements of $C(t_1)$ with integral coefficients is the trivial one.
Consider any vanishing linear combination

$$\sum_{j=1}^{m} k_j ((t_1)) r_{i_{j,1}} r_{i_{j,2}} ... r_{i_{j,n_j}} = 0 \tag{3.5}$$

of pairwise distinct conjugates $((t_1)) r_{i_{j,1}} r_{i_{j,2}} ... r_{i_{j,n_j}}$ of t_1, with coefficients $k_1, ..., k_m \in \mathbb{Z}$. Since $((t_1)) r_1 r_2 r_3 = t_1$, without loss of generality we assume that all the n_j are even. Then, (3.3) tells us that the oriented angle from t_1 to $((t_1)) r_{i_{j,1}} r_{i_{j,2}} ... r_{i_{j,n_j}}$ equals

$$2(\beta_{i_{j,2}} - \beta_{i_{j,1}}) + ... + 2(\beta_{i_{j,n_j}} - \beta_{i_{j,n_j - 1}}) \bmod 2\pi, \tag{3.6}$$

where β_i denotes the oriented angle from e_1 to e_i, namely $\beta_1 = 0, \beta_2 = \pi - \alpha_3$ and $\beta_3 = \pi + \alpha_2 \bmod 2\pi$. This can also be written in the form

$$m_{j,2}\alpha_2 + m_{j,3}\alpha_3 \bmod 2\pi, \tag{3.7}$$

with $m_{j,2}$ and $m_{j,3}$ even integers, hence the scalar product of (3.5) with t_1 gives

$$\sum_{j=1}^{m} k_j \cos(m_{j,2}\alpha_2 + m_{j,3}\alpha_3) = 0, \tag{3.8}$$

where ρ^2, the squared norm of t_1 (and all its conjugates), has been collected as a common factor and canceled. Similarly, the scalar product of (3.5) with the vector obtained by rotating t_1 of $\pi/2$ radians gives

$$\sum_{j=1}^{m} k_j \sin(m_{j,2}\alpha_2 + m_{j,3}\alpha_3) = 0. \tag{3.9}$$

Notice that in the above equations the pairs $(m_{j,2}, m_{j,3})$ are different from each other, in that the conjugates in (3.5) are pairwise distinct. Moreover, possibly after suitable changes of signs in order to have either $m_{j,2} > 0$ or $m_{j,2} = 0$ and $m_{j,3} = 0$, we can collect the (at most two) terms corresponding to opposite pairs. According to 3.1.4, the identities (3.8) and (3.9) hold for every triangle, that is for every $\alpha_2, \alpha_3 > 0$ such that $\alpha_2 + \alpha_3 < \pi$. Therefore, the linear independence over the reals of the complex functions $(x_1, x_2) \mapsto \exp(i(m_1x_1 + m_2x_2))$ with $m_1 > 0$ or $m_1 = 0$ and $m_2 = 0$, allows us to conclude that $k_j = 0$ for every $j = 1, ..., m$. $\square$

In view of the above proof, if τ is a generic triangle then any conjugate $t \in C(t_1)$ can be obtained from t_1 by a rotation of $2m\alpha_2 - 2n\alpha_3$ radians for some (uniquely determined, as τ is typical) integers m and n, hence $t = ((t_1))(r_1r_2)^n(r_1r_3)^m$ according to equations (3.3) and (3.4). Therefore, for a generic triangle τ we can write

$$C(t_1) = \{t_{n,m} = ((t_1))(r_1r_2)^n(r_1r_3)^m,\ n, m \in \mathbb{Z}\}. \tag{3.10}$$

Theorem 3.3.2. *For a generic triangle τ the group G_τ admits the presentation*

$$\langle x_1, x_2, x_3 \mid x_1^2, x_2^2, x_3^2, [w, ((w))(x_1x_2)^n(x_1x_3)^m], n, m \in \mathbb{Z},\ (n, m) \neq (0, 0)\rangle,$$

with $w = (x_1x_2x_3)^2$ and the symbols x_1, x_2, x_3 corresponding to r_1, r_2, r_3, respectively.

Proof. By a standard argument (see [[5], Sec. 10.2]), a presentation of G_τ can be derived from presentations of the groups T_τ and S_τ involved in the exact sequence (3.1). In view of 3.1.6 a presentation of S_τ is given by 3.2.4, while T_τ is free abelian on the set of generators (3.10), according to 3.3.1.
Pulling back the generators s_i of S_τ to the generators r_i of G_τ, the relations $s_i^2 = 1$ still hold in the same form $r_i^2 = 1$, while the relation $(s_1s_2s_3)^2 = 1$ turns into the identity $(r_1r_2r_3)^2 = t_1 = t_{0,0}$. Moreover, based on (3.10), for the generators of T_τ we have

$$t_{n,m} = ((t_{0,0}))(r_1r_2)^n(r_1r_3)^m = (((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^m. \tag{3.11}$$

At this point, to complete the set of relations for G_τ it remains to rewrite in the generators r_i's, by using equation (3.11), the commutators $[t_{n,m}, t_{n',m'}]$ and the equations

$$((t_{n,m}))r_1 = t_{-n+1,-m-1}, \quad ((t_{n,m}))r_2 = t_{-n+2,-m-1}, \quad ((t_{n,m}))r_3 = t_{-n+1,-m}, \tag{3.12}$$

which express in terms of the $t_{n,m}$'s their conjugates by the r_i's. The latter equations could be easily shown to hold, by taking into account equation (3.4) and the commutativity of $SO(2)$, and by using the relations r_i^2 and the trivial identity $(((r_1r_2r_3)^2))r_1r_2r_3 = (r_1r_2r_3)^2$. However, we are going to validate them in a different way.
In fact, the rest of the proof is aimed to see how the rewriting of equations (3.12) in the r_i's, as well the rewriting of the commutators $[t_{n,m}, t_{n',m'}]$ with $(n', m') \neq (n, m)$, can be derived from the relations r_i^2 and the relations

$$[(r_1r_2r_3)^2,\ (((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^m] \tag{3.13}$$

with $n, m \in \mathbb{Z}$ and $(n, m) \neq (0, 0)$, which represent the special commutators $[t_{0,0}, t_{n,m}]$.
We start by observing that the relations r_i^2 imply

$$[r_1r_3, r_1r_2] = r_1r_3r_1r_2r_3r_1r_2r_1 = (((r_1r_2r_3)^2))(r_1r_3)^{-1}, \tag{3.14}$$

which in turn, together with the relation $[(r_1r_2r_3)^2, (((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^m]$ conjugated by $(r_1r_3)^{-1}$, implies

$$(((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^m r_1r_2 = (((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^{m\mp1}r_1r_2(r_1r_3)^{\pm1}.$$

Hence, by increasing/decreasing induction on m, based on the trivial case of $m = 0$,

$$(((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^m(r_1r_2)^{\pm1} = (((r_1r_2r_3)^2))(r_1r_2)^{n\pm1}(r_1r_3)^m$$

for every $m \in \mathbb{Z}$. Finally, by increasing/decreasing induction on n, based on the trivial case of $n = 0$, we get

$$(((r_1r_2r_3)^2))(r_1r_3)^m(r_1r_2)^n = (((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^m \tag{3.15}$$

for every $n, m \in \mathbb{Z}$.
As a consequence of (3.15), the rewriting of any commutator $[t_{n,m}, t_{n',m'}]$ is equivalent up to conjugation to that of $[t_{0,0}, t_{n'-n,m'-m}]$. In fact,

$$[(((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^m, (((r_1r_2r_3)^2))(r_1r_2)^{n'}(r_1r_3)^{m'}]$$

once conjugated by $(r_1r_3)^{-m}(r_1r_2)^{-n}$ becomes

$$[(r_1r_2r_3)^2, (((r_1r_2r_3)^2))(r_1r_2)^{n'}(r_1r_3)^{m'-m}(r_1r_2)^{-n}],$$

and this is equivalent to

$$[(r_1r_2r_3)^2, (((r_1r_2r_3)^2))(r_1r_2)^{n'-n}(r_1r_3)^{m'-m}]$$

by equation (3.15).
Moreover, we obtain the rewriting of the relations (3.12) from the relations r_i^2 and the relations (3.13), by the following chains of equalities, whose last step is based on two applications of equation (3.12):

$$\begin{aligned}(((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^m r_1 &= (((r_1r_2r_3)^2))r_1(r_1r_2)^{-n}(r_1r_3)^{-m}\\ &= (((r_1r_2r_3)^2))r_1r_2r_3r_1(r_1r_2)^{-n}(r_1r_3)^{-m}\\ &= (((r_1r_2r_3)^2))(r_1r_2)^{-n+1}(r_1r_3)^{-m-1};\end{aligned}$$

$$\begin{aligned}(((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^m r_2 &= (((r_1r_2r_3)^2))r_1(r_1r_2)^{-n}(r_1r_3)^{-m}r_1r_2\\ &= (((r_1r_2r_3)^2))r_1r_2r_3r_1(r_1r_2)^{-n}(r_1r_3)^{-m}r_1r_2\\ &= (((r_1r_2r_3)^2))(r_1r_2)^{-n+2}(r_1r_3)^{-m-1};\end{aligned}$$

$$\begin{aligned}(((r_1r_2r_3)^2))(r_1r_2)^n(r_1r_3)^m r_3 &= (((r_1r_2r_3)^2))r_1(r_1r_2)^{-n}(r_1r_3)^{-m}r_1r_3\\ &= (((r_1r_2r_3)^2))r_1r_2r_3r_1(r_1r_2)^{-n}(r_1r_3)^{-m}r_1r_3\\ &= (((r_1r_2r_3)^2))(r_1r_2)^{-n+1}(r_1r_3)^{-m}.\end{aligned}$$

This concludes the proof. □

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