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3-MANIFOLDS AND PULLBACKS OF BRANCHED COVERINGS

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ABSTRACT

We prove that any closed orientable 3-manifold can be represented as the pullback space of two 3-fold simple branched coverings of S^3 by itself. By using covering moves, we also prove that such two coverings can be chosen equivalent to arbitrarily given ones.

Keywords: Pullback, special branched covering, covering move, 3-manifold.

Introduction

In [1], H. M. Hilden, M. T. Lozano and J. M. Montesinos proved that any closed orientable 3-manifold is the pullback of any fixed irregular 3-fold branched covering $p : S^3 \to S^3$ by a suitable smooth map $g : S^3 \to S^3$.

By using a different construction, we show that the map g can be chosen to be a 3-fold simple branched covering, if the covering p is simple. Moreover, the covering moves introduced in [2] are applied for getting such a covering g equivalent to any given 3-fold simple branched covering of S^3 by itself.

This allows us to effectively represent 3-manifolds as pullbacks of branched coverings of S^3 and makes likely the introduction of a calculus for this representations (cf. note 2 of [1]). A nice example of such a way to represent 3-manifolds is proposed at the end of this paper.

The result mentioned above is stated in theorem 8, all the rest of the paper is devoted to establish facts we need for its proof.

0. Preliminaries

By a *branched covering* of the closed 3-manifold N by the closed 3-manifold M, we mean a non-degenerate piecewise linear map $p : M \to N$, which is a finite ordinary covering over the complement of a *branch link* $L_p \subset N$.

The *monodromy* of p is the monodromy $\omega_p : \pi_1(N - L_p) \to \Sigma_d$ of the ordinary covering $c_p = p_| : M - p^{-1}(L_p) \to N - L_p$ with respect to any fixed base point. We say that p is *simple* iff its monodromy w_p sends all the meridians of L_p into transpositions.

It is well known that a branched covering $p : M \to N$ is uniquely determined (up to homeomorphisms of M) by its branch link L_p and its monodromy w_p.

By a *splitting set* for p we mean a two-dimensional subpolyhedron $S \subset N$ containing L_p, such that $N - S$ is connected and w_p is trivial on $\pi_1(N - S)$. The covering manifold M can be constructed by taking d copies of N split along S and gluing them together according to the monodromy w_p.

Two branched coverings $p : M \to N$ and $p' : M' \to N'$ are called *equivalent* iff there exist two homeomorphisms $h : N \to N'$ and $k : M \to M'$ making the following diagram commutative.

$$\begin{array}{ccc} M & \xrightarrow{k} & M' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ N & \xrightarrow[h]{} & N' \end{array}$$

It is a standard fact that: p and p' are equivalent if there exists a homeomorphism $h : N \to N'$, such that $h(L_p) = L_{p'}$ and $w_{p'}\, h_* = w_p$ where $h_* : \pi_1(N - L_p) \to \pi_1(N' - L_{p'})$ is the homomorphism induced by the restriction of h.

Finally, by $B_i^2 \subset R^2 \subset R^3 \subset R^3 \cup \{\infty\} \cong S^3$ we will denote the plane disk of center $(2i, 0)$ and radius $1/2$, and by $S_i^1 = \mathrm{Bd}\, B_i^2$ its boundary.

1. Special Branched Coverings

In this section we prove an enhanced version of theorem 1 of [1], about representation of 3-manifolds as special branched covers of S^3. This will be the starting point for getting our main goal.

Definition 1. By a *special branched covering* we mean a branched covering $p : M \to N$ which admits as a splitting set a locally flat orientable surface F such that $\mathrm{Bd}\, F = L_p$.

As an immediate consequence of the definition, once fixed orientations of F and N, there exists a unique locally constant application $\varphi : F \to \Sigma_d$, such that $\omega_p(\alpha) = \varphi(x_1)^{\varepsilon_1} \dots \varphi(x_n)^{\varepsilon_n}$, where: α is any element of $\pi_1(N - L_p)$, $x_1, \dots, x_n$ is the ordered sequence of the intersections with F of an arbitrary loop representing α and transversal to F, and ε_i is the sign of the intersection x_i with respect to the given orientations.

Given any compact oriented locally flat surface $F \subset N$ with no closed components, any orientation of N and any locally constant application $\varphi : F \to \Sigma_d$, we will denote by $p_{F,\varphi}$ the special branched covering whose branch link is $\mathrm{Bd}\, F$ and whose monodromy is defined as above. The corresponding covering manifold will denoted by $M(F, \varphi)$.

Then, $M(F,\varphi)$ can be obtained from N as follows: 1) split N along $\operatorname{Int} F$ and denote by x^- and x^+ the two points coming from the point $x \in \operatorname{Int} F$, according to the given orientations; 2) take d copies of the result, and denote by x_i the i-th copy of the point x; 3) glue them by identifying x_i^- with $x^+_{\varphi(x)(i)}$.

We remark that $p_{F,\varphi}$ and $M(F,\varphi)$ do not depend on the orientations if $\varphi(F)$ consists only of transpositions, that is if the covering is simple.

By [1], any closed orientable 3-manifold M can be represented as a 3-fold simple special branched cover of S^3. More precisely, there exist two disjoint locally flat orientable surfaces $F_1, F_2 \subset S^3$, such that $M = M(F,\varphi)$ where $F = F_1 \cup F_2 \subset S^3$ and φ is defined by $\varphi(F_1) = (1\,2)$ and $\varphi(F_2) = (2\,3)$.

Now, denoting by $M(F_1, F_2)$ such a branched cover of S^3, we will show that the surfaces F_1 and F_2 can be chosen with some further property.

Proposition 2. *Any closed orientable* 3*-manifold* M *can be represented as a* 3*-fold simple special branched cover* $M(F_1,F_2)$ *of* S^3*, where* $\operatorname{Bd} F_1$ *has one or two components and* $\operatorname{Bd} F_2$ *is connected.*

Proof. We start as in [1], by considering a representation of M by surgery on a framed link $L \subset S^3$ of class 3 in the sense of [3]. More precisely, the link L is the union of two sublinks L' and L'' embedded in $R^3 \subset R^3 \cup \{\infty\} \cong S^3$ in the following way (cf. figure 1, [3] and [1]):

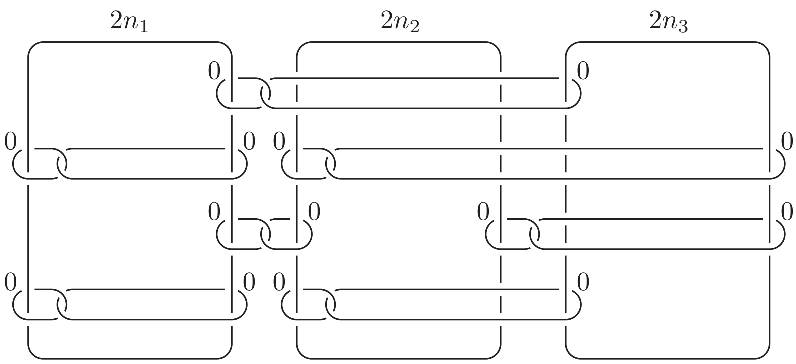

Figure 1.

a) L' is the union of an arbitrary number of rectangular components, all lying in the projection plane of the diagram and having even framing;

b) L'' is the union of an arbitrary number of separated pairs of once simply linked meridional circles for components of L', all projecting as topological circles in the diagram and having framing zero.

Now, by using moves of type 3.3 (a) of [3], we modify each pair of contiguous components of L' as described in figure 2, in such a way that the top of L' appears to be as in figure 3. Of course, if L' has an even number of components, all them are involved in this process, otherwise the last one is fixed.

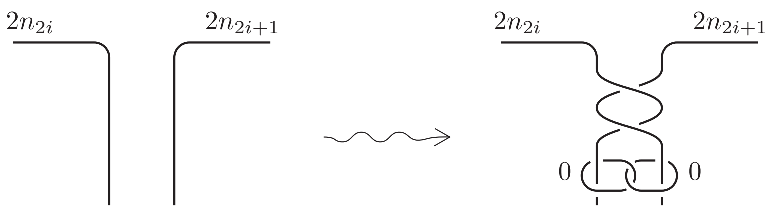

Figure 2.

Then, we include all the new components appearing in the process (cf. figure 2) into L'', and move this extended L'' down, in order to get a diagram as in figure 3. We observe that here each component of L'' is symmetric with respect to the axis α'', and also each component of L' can be thought symmetric with respect to the axis α'.

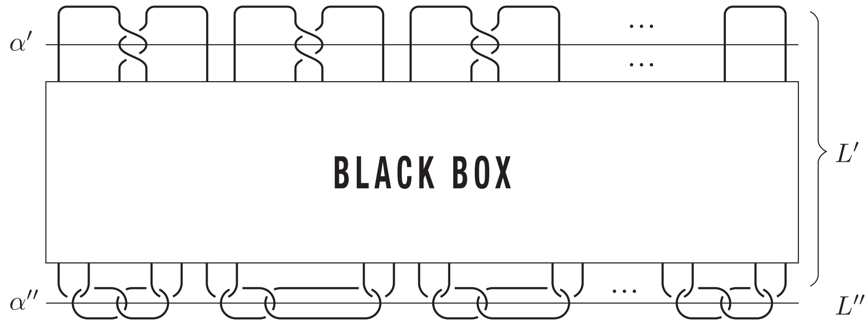

Figure 3.

Finally, we conclude the proof as in [1], by applying the algorithm introduced in [4]. In this way, we get M as a cover of S^3 branched over the boundary of two orientable 2-disks with twisted bands parallel to the part of L lying between α' and α'' (cf. figure 4). It is easy to check that such branch link satisfy the required connectedness properties. □

It is not clear whether proposition 2 can be improved by obtaining $\mathrm{Bd}\, F_1$ and $\mathrm{Bd}\, F_2$ both connected. Anyway, this is irrelevant for our aims.

2. Pulling back branched coverings

Now, by using results of [5], we show how to realize any special branched covering of S^3 as the pullback of a "very" special branched covering of S^3 by a simple branched covering of S^3.

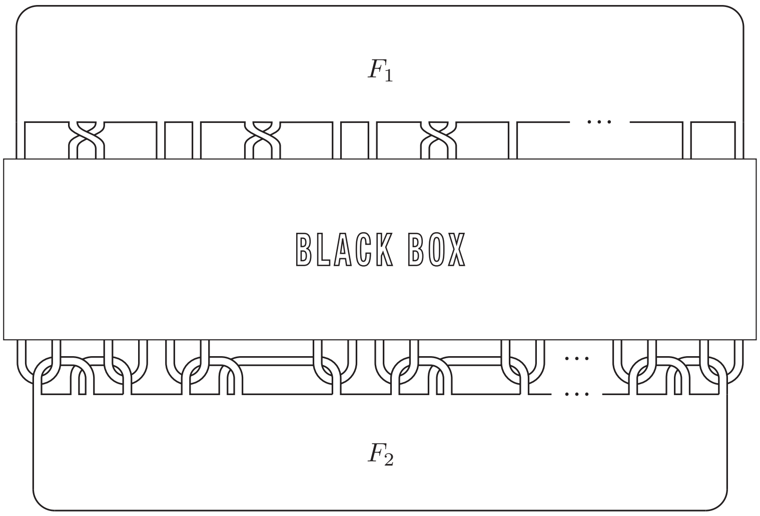

Figure 4.

Lemma 3. *Let $p : P \to N$ and $q : Q \to N$ be branched coverings such that $L_p \cap L_q = \varnothing$, then the maps p' and q' in the following pullback are branched covering, such that:*
a) *$d(p') = d(q)$ and $d(q') = d(p)$;*
b) *$L_{p'} = p^{-1}(L_q)$ and $L_{q'} = q^{-1}(L_p)$;*
c) *$\omega_{p'} = \omega_q\, p_*$ and $\omega_{q'} = \omega_p\, q_*$, where p_* and q_* are the homomorphisms induced on the fundamental groups by the obvious restrictions of p and q.*

$$\begin{array}{ccc} M & \xrightarrow{p'} & P \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle p} \\ Q & \xrightarrow[q]{} & N \end{array}$$

Proof. By symmetry, we can limit ourselves to deal with p'. First of all, we observe that the result trivially holds for p', if q is an ordinary covering, that is if $L_q = \varnothing$. So, by restricting all the construction over $N - L_q$, we get that p' is an ordinary covering with the right properties over $P - p^{-1}(L_q)$. Then, it remains to check that: p is non-degenerate, $p^{-1}(L_q)$ is a link in N and M is a manifold. This easily follows from the fact that p is transversal with respect to L_q (since $L_p \cap L_q = \varnothing$). $\square$

We note that, if the branch links of p and q are not disjoint, their pullback M is not necessarily a manifold (for example, think of p and q as double coverings of B_2 branched over the same point).

In the case of $L_p = L_q$, an alternative way for pulling back branched coverings has been used by Hempel in [6]: first pull back the ordinary coverings c_p and c_q, then complete to branched coverings. Lemma 3 says that, if $L_p \cap L_q = \varnothing$, this construction gives nothing else but the pullback of p and q.

Lemma 4. *Given any closed orientable 3-manifold M, any compact orientable locally flat surface $F \subset M$ with no closed components, any decomposition $F = F_1 \cup \ldots \cup F_k$ into unions of components, and any integer $n \geq \max\{3, \#$ of components of $\mathrm{Bd}\, F_i$ for $i = 1, \ldots, k\}$, there exists a n-fold simple branched covering $p : M \to S^3$ such that $L_p \cap S^1_i = \varnothing$ and $p^{-1}(B^2_i) = F_i$ for $i = 1, \ldots, k$.*

Proof. By the proof of theorem (10.1) and remark (10.3) of [5], there exist n-fold simple branched coverings $p_i : N(F_i) \to N(B^2_i)$, where $N(-)$ denotes sufficiently small regular neighborhoods, such that the branch link of p_i is disjoint from S^1_i and $p^{-1}(B^2_i) = F_i$, for $i = 1, \ldots, k$. Now, by adding 1-handles, we can join them, to form a branched covering $p' : D \to C$, where $D \subset M$ and $C \subset S^3$ is a 3-cell containing $\cup_i N(B^2_i)$. Moreover, by using lemma (6.1) of [5], we can easily assume that $\mathrm{Bd}\, D$ is connected, so that we can extend p' to a branched covering $p : M \to S^3$ as required, by corollary (6.3) of [5]. □

Proposition 5. *Given any d-fold special branched covering $p_{F,\varphi} : M \to N$, let $\{\sigma_1, \ldots, \sigma_k\} = \varphi(F)$ and $F_i = \varphi^{-1}(\sigma_i)$. Then, for any $n \geq \max\{3, \#$ of components of $\mathrm{Bd}\, F_i$ for $i = 1, \ldots, k\}$ there exists a n-fold simple branched covering $q : N \to S^3$ with $L_q \cap \mathrm{Bd}\, G = \varnothing$, giving the following pullback, where $G = B^2_1 \cup \ldots \cup B^2_k$ and $\psi(B^2_i) = \sigma_i$.*

$$\begin{array}{ccc} M & \longrightarrow & S^3 \\ {\scriptstyle p_{F,\varphi}}\big\downarrow & & \big\downarrow{\scriptstyle p_{G,\psi}} \\ N & \underset{q}{\longrightarrow} & S^3 \end{array}$$

Proof. Let q be a branched covering given by lemma 4. It is easy to realize that q can be chosen in order to preserve all the relevant orientations involved in defining $p_{F,\varphi}$ and $p_{G,\psi}$. Then, by lemma 3, the pullback of the two coverings q and $p_{G,\psi}$ gives us a branched covering $p'_{G,\psi} : M \to N$ with the same branch link and the same monodromy as $p_{F,\varphi}$. □

3. Pulling back covering moves

In this section we prove that covering moves, relating 3-fold simple branched coverings of S^3 which represent the same 3-manifold, can be pulled back by suitable branched coverings.

In [2] it is proved that, if $p : P \to S^3$ and $q : Q \to S^3$ are 3-fold simple branched coverings of S^3 by two closed 3-manifolds, then: $P \cong Q \iff p$ and q can be related (up to equivalence) by a finite sequence of the moves shown in figure 5.

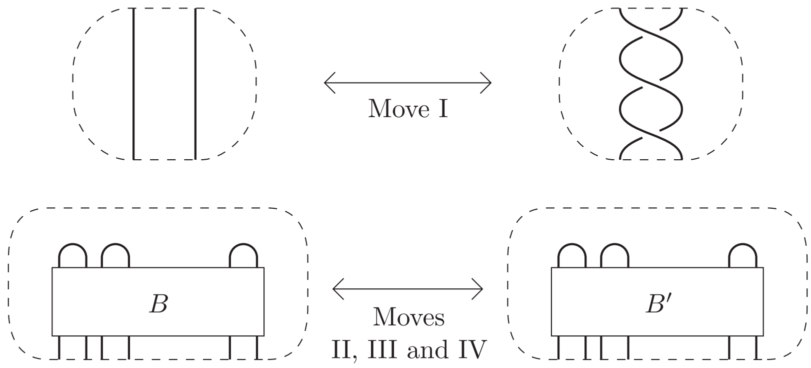

Figure 5.

Here, coverings are represented by means of diagrams of their branch links, whose bridges must be thought labelled with transpositions giving the monodromy of the corresponding meridians. The moves change the coverings only over the 3-cells bounded by the dashed curves, and can be performed iff the monodromies inside these 3-cells are suitable.

The 3-cells considered in moves II, III and IV are the complements of the boxes denoted by L' in figure 7 of [2]. Although not explicitely stated, it is clear from the proof of the Equivalence theorem of [2], that the links denoted by L in the same figure can be assumed to be braids closed at the top as plats. Then, we can think of B and B' in our figure 5 as suitable braids.

Proposition 6. *Given the pullback*

$$\begin{array}{ccc} M & \longrightarrow & N \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ S^3 & \xrightarrow[q]{} & S^3 \end{array}$$

where p and q are 3-fold simple branched coverings, for any other 3-fold simple branched covering $p' : N \to S^3$, there exists a branched covering q' equivalent to q, making the following diagram a pullback.

$$\begin{array}{ccc} M & \longrightarrow & N \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p'} \\ S^3 & \xrightarrow[q']{} & S^3 \end{array}$$

Proof. Of course, we can limit ourselves to deal with the case of p' obtained from p by just one move. Let C be the 3-cell involved in the move, and $h : S^3 \to S^3$

be a homeomorphism such that: 1) h fixes L_p; 2) $h(L_q) \subset S^3 - C$; 3) h is isotopic to the identity mod L_p. Such a homeomorphism trivially exists for the move I, and can be constructed by following the isotopy corresponding to the braids in figure 5, for the other moves. Then, we put $q' = h\,q$ and consider the following pullback.

$$\begin{array}{ccc} M & \longrightarrow & N \\ {\scriptstyle r}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ S^3 & \underset{q'}{\longrightarrow} & S^3 \end{array}$$

We remark that the pullback space is again M. This is because, by lifting h by means of the covering q', we get a homeomorphism k of S^3 onto itself, which gives us the equivalence between r and the corresponding covering in the original diagram. Since $L_{q'} = h(L_q)$ does not meet C, we have that $q'^{-1}(C)$ is the disjoint union of three 3-cells $C_1 \cup C_2 \cup C_3$. Hence, the move performed on p in order to get p' lifts, by means of q', to three moves of the same type on r taking place in the 3-cells C_i. Then, the required pullback diagram can be obtained from the last one, by performing these moves. □

4. 3-manifolds as pullback covers

At this point, we are in position to prove our main result, just by collecting all the facts proved in the previous sections. In the following lemma, we denote by b_3 the 3-fold simple special covering of S^3 by itself, representing S^3 as $M(B_1^2, B_2^2)$ (cf. section 1).

Lemma 7. *For any closed orientable 3-manifold M there exists a 3-fold simple branched covering $q : S^3 \to S^3$ making the following diagram a pullback.*

$$\begin{array}{ccc} M & \longrightarrow & S^3 \\ \Big\downarrow & & \Big\downarrow{\scriptstyle b_3} \\ S^3 & \underset{q}{\longrightarrow} & S^3 \end{array}$$

Proof. The branched covering q can be immediately obtained, by applying proposition 5 to any representation of M as a 3-fold special branched cover of S^3 given by proposition 2. □

Theorem 8. *For any closed orientable 3-manifold M and given any two 3-fold simple branched coverings $p, q : S^3 \to S^3$, there exists a pullback diagram*

$$\begin{array}{ccc} M & \longrightarrow & S^3 \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ S^3 & \underset{q'}{\longrightarrow} & S^3 \end{array}$$

where q' is equivalent to q.

Proof. We begin with a pullback diagram as in lemma 7. Then, we apply proposition 6 in order to get q on the bottom edge of the diagram, changing b_3 to an equivalent covering. Finally, we apply again proposition 6 in order to get p on the right edge of the diagram, changing q into q'. $\square$

We conclude this paper, by proposing the following way to represent 3-manifolds as pullbacks of branched coverings of the 3-sphere. Let $L_1, L_2 \subset S^3$ be any two disjoint links, both consisting of two unknotted and unlinked components. Then, there are two 3-fold simple special coverings p_1 and p_2 branched over L_1 and L_2 uniquely determined up to equivalence. By considering the pullback of these two coverings we get a closed orientable 3-manifold $M(L_1, L_2)$, which is uniquely determined up to homeomorphism. By theorem 8, there exists such a nice representation for any closed orientable 3-manifold.

References

[1] H. M. Hilden, M. T. Lozano and J. M. Montesinos, *All three-manifolds are pullbacks of a branched covering S^3 to S^3*, Trans. Amer. Math. Soc. **279** (1983), 729–735.

[2] R. Piergallini, *Covering moves*, Trans. Amer. Math. Soc. **325** (1991), 903–920.

[3] J. S. Birman and J. Powell, *Special Representations for 3-Manifolds*, Geometric Topology, J. C. Cantrell ed., Academic Press 1979, 23–51.

[4] J. M. Montesinos, *A note on 3-fold branched coverings of S^3*, Math. Proc. Camb. Phil. Soc. **88** (1980), 321–325.

[5] I. Berstein and A. L. Edmonds, *On the construction of branched coverings of low-dimensional manifolds*, Trans. Amer. Math. Soc. **247** (1979), 87–124.

[6] J. Hempel, *Branched covers over strongly amphicheiral links*, Topology **29** (1990), 247–255.