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COVERING HOMOTOPY 3-SPHERES

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ABSTRACT – We prove the following theorem: for any closed orientable 3-manifold M and any homotopy 3-sphere Σ, there exists a simple 3-fold branched covering $p : M \to \Sigma$.

We also propose the conjecture that, for any primitive branched covering $p : M \to N$ between orientable 3-manifolds, $g(M) \geq g(N)$, where g denotes the Heegaard genus. By the above mentioned result, the genus 0 case of such conjecture is equivalent to the Poincaré conjecture.

Introduction

An *n-fold branched covering* $p : M \to N$ between connected pl 3-manifolds, is a non-degenerate pl map for which there exist two links $S(p) \subset M$ and $B(p) \subset N$ (the *branch link*), such that the restriction $p_{|M-S(p)} : M - S(p) \to N - B(p)$ is an ordinary n-fold covering.

The branched covering p can be described in terms of the monodromy ω of $p_{|M-S(p)}$ (with respect to any base-point), which represents $\pi_1(N - B(p))$ into Σ_n, the symmetric group of degree n.

We say that p is *simple* if $\omega(\mu)$ is a transposition for each meridian μ around $B(p)$. Moreover, p is *primitive* if it cannot be factored as a branched covering followed by an ordinary covering, that is if it induces a surjection between fundamental groups.

A well known theorem proved by Hilden [2], Hirsch [3] and Montesinos [5], says that every closed orientable 3-manifold can be represented as a 3-fold simple branched cover of S^3.

In this paper we adapt the proof of that theorem given in [6], in order to prove that in fact S^3 can be replaced with any homotopy 3-sphere Σ.

In particular, we have that any homotopy 3-sphere is covered by S^3. This fact allows us to see the Poincaré conjecture as a special case of the much more general conjecture that primitive branched coverings cannot increase the Heegaard genus.

1. Branched coverings of homotopy 3-spheres

The following theorem is the version for homotopy 3-spheres, of the Hilden-Hirsch-Montesinos's theorem on branched coverings of S^3 mentioned in the introduction. The proof of the theorem follows essentially the same line of [6] (cf. also [7]).

Theorem. *For any closed orientable 3-manifold M and any homotopy 3-sphere Σ, there exists a simple 3-fold branched covering $p : M \to \Sigma$.*

Proof. Let Σ be a homotopy 3-sphere, and $p_\Sigma : \Sigma \# \Sigma \# \Sigma \to \Sigma$ a simple 3-fold covering branched over $B(p_\Sigma)$ = two unlinked trivial knots, with monodromy respectively $(1\,2)$ and $(2\,3)$.

If $C \subset \Sigma$ is a 3-cell and $\Gamma = \mathrm{Cl}(\Sigma - C)$, then p_Σ is equivalent to $p_C \cup_{\mathrm{Bd}} p_\Gamma$, where $p_C : C \to C$ is the simple 3-fold branched covering shown in Figure 1, and $p_\Gamma : \Gamma_1 \#_{\mathrm{Bd}} \Gamma_2 \#_{\mathrm{Bd}} \Gamma_3 \to \Gamma$ is obtained by replacing a 3-cell in the interior of C with a copy of Γ, as shown in Figure 2.

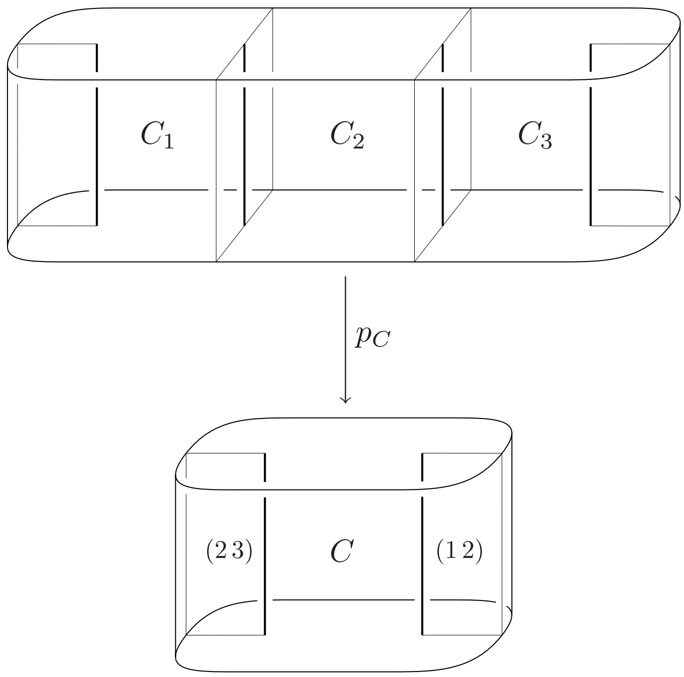

Figure 1.

Now, let M be a closed orientable 3-manifold and $L_1, L_2, L_3 \subset \operatorname{Int} \Gamma$ be disjoint links, such that M can be obtained from Σ by surgery on L_2, and there are surgeries on L_1 and L_3 giving S^3.

Since Γ is simply connected, L_i bounds a singular union of disks $F_i \subset \operatorname{Int} \Gamma$ that can be assumed, by using a well-known piping technique, to have only clasp singularities (cf. Figure 3), for $i = 1, 2, 3$. Moreover, since such F_i's separately collapse into graphs, we can also easily assume that they are disjoint from each other.

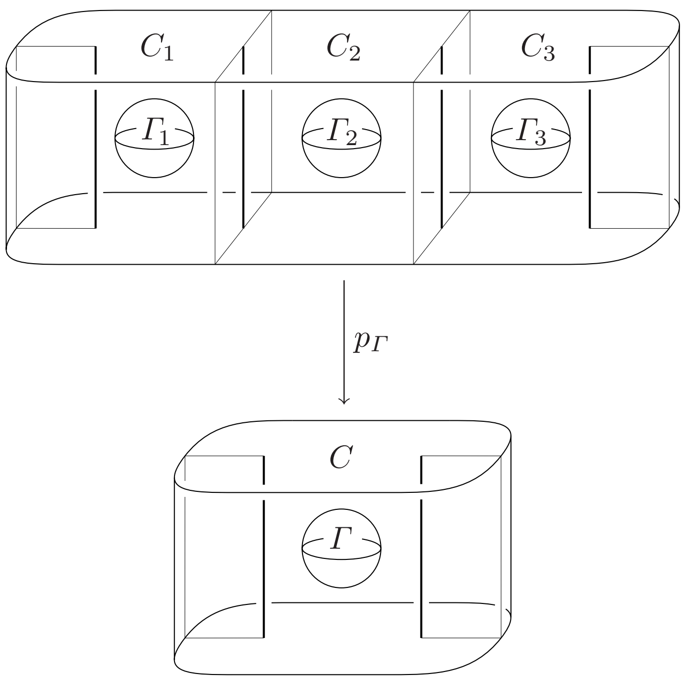

Figure 2.

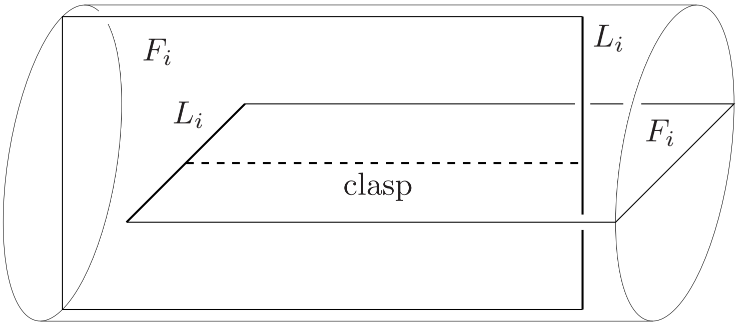

Figure 3.

In Figure 3 is represented (up to homeomorphism) a neighborhood of a clasp of F_i. Now, we can change the surgery link L_i without changing the resulting manifold, by using the operations introduced by Kirby in [4]. In particular we perform the operation described in Prop. 1A of [4], inside the neighborhood in Figure 3, in order to eliminate the clasp, as shown in Figure 4.

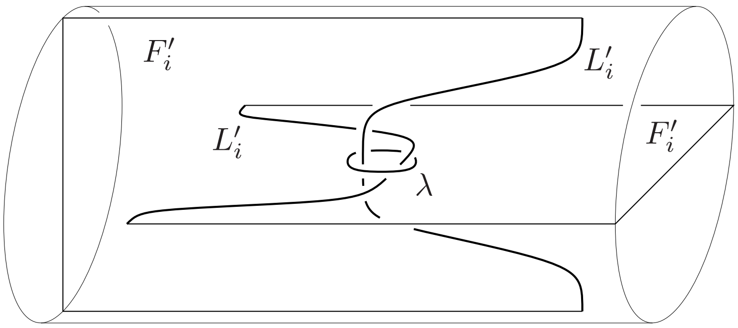

Figure 4.

Let L_i' be the modified link L_i after all the clasps of F_i have been removed as said above, and L_i'' be the union of all the trivial loops λ added performing such modifications (cf. Figure 4), for $i = 1, 2, 3$. These new links have the following properties:

(1) M can be obtained from Σ by surgery on $L_2' \cup L_2''$;
(2) S^3 can be obtained from Σ by surgery on $L_1' \cup L_1''$ and $L_3' \cup L_3''$;
(3) L_i' and L_i'' are trivial links in $\Gamma - (L_j' \cup L_j'' \cup L_k' \cup L_k'')$, that is they respectively bound two non-singular union of disks F_i' and F_i'' in $\Gamma - (L_j' \cup L_j'' \cup L_k' \cup L_k'')$, for $\{i, j, k\} = \{1, 2, 3\}$.

Now, let $\alpha_i', \alpha_i'' \subset C$ be the arcs between $\mathrm{Bd}\,\Gamma$ and the branch link $B(p_\Sigma)$ of the covering p_Σ, shown in Figure 5.

We can assume, up to an ambient isotopy of Σ, that each component G of F_i' (resp. F_i'') meets $B(p_\Sigma)$ exactly in a small arc of its boundary, in such a way that $\mathrm{Cl}(G - \Gamma)$ consists of a band parallel to the arc α_i' (resp. α_i''), for $i = 1, 2, 3$.

Then $A = \mathrm{Cl}(\cup_{i=1,2,3}(L_i' \cup L_i'') - B(p_\Sigma))$ consists of finitely many arcs in Σ, whose end-points are in $B(p_\Sigma)$. Moreover, by property 3, $p_\Sigma^{-1}(A)$ is isotopically equivalent to $\widetilde{L}_1 \cup \widetilde{L}_2 \cup \widetilde{L}_3 \cup \widetilde{A} \subset \Sigma \# \Sigma \# \Sigma$, where: $\widetilde{L}_i$ is a copy of $L_i' \cup L_i''$ inside Γ_i for $i = 1, 2, 3$, and $\widetilde{A}$ consists of finitely many arcs (cf. Figure 5).

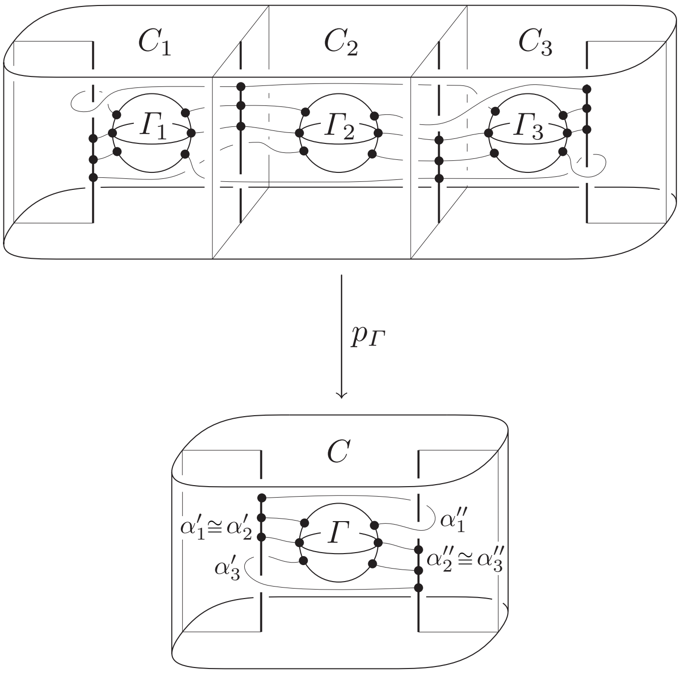

Figure 5.

Now, following [6], it is easy to prove, that the surgeries of properties 1 and 2, performed on the links $\widetilde{L}_i$, can be realized lifting (by means of p_Σ) trivial surgeries (= removing 3-cells from Σ and sewing them back differently). In such a way we get $M \cong S^3 \# M \# S^3$ as a branched covering of Σ. $\square$

We conclude this section with a corollary, which perhaps might be useful in order to prove the Poincaré conjecture. In the next section we will discuss a conjecture in this direction.

Corollary. *For any homotopy 3-sphere Σ there exists a simple 3-fold branched covering $p : S^3 \to \Sigma$.*

2. Branched coverings and Heegaard genus

In this section we propose and briefly discuss a conjecture about Heegaard genus of branched coverings, which is related to various other conjectures about 3-manifolds.

First of all, we observe that, in the light of the corollary above, the Poincaré conjecture is equivalent to the following one: if $p : S^3 \to M$ is a primitive branched covering, then $M \cong S^3$. By the positive solution of the Smith conjecture ([8]), this is known to be true when p is cyclic.

Now, as a generalization of such a formulation of the Poincaré conjecture, we conjecture that primitive branched coverings cannot increase the Heegaard genus. More precisely, we propose the following:

Conjecture. *If $p : M \to N$ is a primitive branched covering between closed orientable 3-manifolds, then $g(M) \geq g(N)$, where g denotes the Heegaard genus.*

Of course, this conjecture becomes trivially true if the Heegaard genus is replaced by the rank of the fundamental group of the manifolds. So it can be thought as a weakening of the Waldhausen conjecture about the coincidence of the rank with the Heegaard genus.

In [1], Boileau and Zieschang proved that the Waldhausen conjecture is true for most of the closed orientable Seifert manifolds. Then, the same holds for our conjecture.

In the same paper, Boileau and Zieschang also give a counterexample to the Waldhausen conjecture, proving that, for the Seifert manifolds $N = S(0; e_0; 1/2, 1/2, 1/2, \beta/2\lambda + 1)$, with $\lambda > 0$, $\gcd(\beta, 2\lambda + 1) = 1$ and $e_0 \neq \pm 1/2(2\lambda + 1)$, one has $\mathrm{rk}(N) = 2$ and $g(N) = 3$. Then a possible counterexample to our conjecture, could be given by a genus 2 primitive branched covering of such an N.

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