Published in: Rend. Ist. Mat. Univ. Trieste, Suppl. 1, XXXII (2001), 193–219

LIFTING BRAIDS

M. Mulazzani
Dipartimento di Matematica
Università di Bologna – Italy
mulazza@dm.unibo.it

R. Piergallini
Dipartimento di Matematica e Fisica
Università di Camerino – Italy
pierg@camserv.unicam.it

Abstract

In this paper we study the homeomorphisms of B^2 that are liftable with respect to a simple branched covering. Since any such homeomorphism maps the branch set of the covering onto itself and liftability is invariant up to isotopy fixing the branch set, we are dealing in fact with liftable braids. We prove that the group of liftable braids is finitely generated by liftable powers of half-twists around arcs joining branch points. A set of such generators is explicitly determined for the special case of branched coverings $B^2 \to B^2$. As a preliminary result we also obtain the classification of all the simple branched coverings of B^2.

Key words and phrases: branched covering of the disk, liftable homeomorphism, liftable braid.

2000 Mathematics Subject Classification: 57M12.

Introduction

A continuous map $p : F \to G$ between compact surfaces with G connected and oriented is a *branched covering* iff it is a local homeomorphism near $\mathrm{Bd}\, F$ and any point $x \in \mathrm{Int}\, F$ has a neighborhood $U \subset \mathrm{Int}\, F$ such that the restriction $p_{|U} : U \to p(U)$ is topologically equivalent to the complex map $z \mapsto z^{d_x}$, for a uniquely determined positive integer d_x, the *local order* of p at x. In particular, we have $p(\mathrm{Int}\, F) = \mathrm{Int}\, G$ and $p(\mathrm{Bd}\, F) = \mathrm{Bd}\, G$.

Given a branched covering $p : F \to G$, we denote by $S_p \subset \mathrm{Int}\, F$ the (finite) set of the *singular points* of p, that is the points $x \in \mathrm{Int}\, F$ such that $d_x > 1$, and by $B_p = \{P_1, \dots, P_n\} \subset \mathrm{Int}\, G$ the set of *branch points* of p, defined by $B_p = p(S_p)$. Then, the restriction $p_| : F - p^{-1}(B_p) \to G - B_p$ is an ordinary covering with d sheets, where $d = d(p)$ is the *order* of p. The orientation of G can be lifted to an orientation of F which makes $p : (F, \mathrm{Bd}\, F) \to (G, \mathrm{Bd}\, G)$ a map of positive degree $d(p)$. *We assume F oriented in this way.*

Since $p_|$ uniquely determines p, by fixing a base point $* \in G - B_p$ and numbering the fiber $p^{-1}(*)$, we can represent p by means of the monodromy

$\varphi_p : \pi_1(G - B_p, *) \to \Sigma_d$ of the ordinary covering $p_|$, where Σ_d is the permutation group on $\{1, \dots, d\}$. We call φ_p the *monodromy* of p. In order to simplify the notation, we write φ in place of φ_p, when there is no risk of confusion. Because of the choices of $*$ and of the numbering of $p^{-1}(*)$, the monodromy is defined only up to inner automorphisms of Σ_d.

A branched covering p is called *simple* iff it maps S_p injectively onto B_p and $d_x = 2$ for any $x \in S_p$. This means that the monodromy of a small simple loop around any branch point is a transposition.

Two branched coverings $p : F \to G$ and $p' : F' \to G'$ are *equivalent* iff there exist *orientation preserving* homeomorphisms $h : G \to G'$ and $k : F \to F'$ such that $p'k = h\,p$. Of course, in this case we have $d(p) = d(p')$, $h(B_p) = B_{p'}$ and $k(S_p) = S_{p'}$. Now, it turns out that the existence of a lifting $k : F \to F'$ of a given homeomorphism $h : G \to G'$ such that $h(B_p) = B_{p'}$ depends only on the existence of a lifting of the restriction $h_| : G - B_p \to G' - B_{p'}$. Then, by the classical theory of ordinary covering, we get the following criterion.

Lifting theorem. *A homeomorphism $h : G \to G'$ has a lifting $k : F \to F'$ with respect to the branched coverings $p : F \to G$ and $p' : F' \to G'$ of the same order d iff $h(B_p) = B_{p'}$ and there exists an inner automorphism ε of Σ_d such that $\varphi_{p'} h_* = \varepsilon\,\varphi_p$, where $h_* : \pi_1(G - B_p, *) \to \pi_1(G' - B_{p'}, *')$ is the isomorphism induced by the restriction of h. In this case ε is given by the conjugation by $\sigma = \nu' k\,\nu^{-1} \in \Sigma_d$, where $\nu : p^{-1}(*) \to \{1, \dots, d\}$ and $\nu' : p'^{-1}(*') \to \{1, \dots, d\}$ are the numberings of the fibers $p^{-1}(*)$ and $p'^{-1}(*')$, with $*' = h(*)$, inducing the monodromies φ_p and $\varphi_{p'}$.*

As an immediate consequence of this lifting theorem, we have an equivalence criterion for branched coverings in terms of their branch sets and monodromies.

Equivalence theorem. *Two branched coverings $p : F \to G$ and $p' : F' \to G'$ of the same order d are equivalent iff there exist an orientation preserving homeomorphism $h : G \to G'$ and an inner automorphism ε of Σ_d such that $h(B_p) = B_{p'}$ and $\varphi_{p'} h_* = \varepsilon\,\varphi_p$.*

The classification of the simple branched coverings of S^2 up to equivalence is classical and well known. In [6] and [7], Gabai and Kazez extended such classification to all the closed surfaces. The following Theorem A, giving a classification of the simple branched coverings of B^2, is stated without proof in [2]. In Section 1 we give a proof of Theorem A, by providing a canonical way of representing branched coverings of B^2. We need such canonical representation in order to get our main result about liftable braids.

Given a simple branched covering $p : F \to B^2$ of order d, we fix a base point $* \in S^1$ and a numbering of $p^{-1}(*)$. Then, we define *total monodromy* of p to be the permutation $\varphi_p(\omega) \in \Sigma_d$, where ω is the clockwise oriented simple loop supported by S^1. Moreover, we denote by $\Omega(p)$ the conjugation class of $\varphi_p(\omega)$ in Σ_d, which is uniquely determined by p (actually by the restriction of p over S^1). Now we are in position to state the classification theorem.

Theorem A. *Two connected simple branched coverings $p : F \to B^2$ and $p' : F' \to B^2$ are equivalent iff they have the same order d, the same number n of branch points and $\Omega(p) = \Omega(p')$.*

Since $\Omega(p)$ is the class of d-cycles of Σ_d for any simple branched covering $p : F \to B^2$ with $\mathrm{Bd}\, F$ connected, by the Riemann-Hurwicz formula we easily get the following corollary.

Corollary. *For every compact connected orientable surface F with connected boundary and for every integer $n \geq 2 - \chi(F)$ there exists a unique (up to equivalence) simple covering $p : F \to B^2$ with n branch points.*

Given an orientable surface S and a closed subset $C \subset S$, we denote by $\mathcal{H}(S)$ the group of all the *orientation preserving* homeomorphisms of S onto itself and by $\mathcal{H}(S, C) \subset \mathcal{H}(S)$ the subgroup consisting of all the $h \in \mathcal{H}(S)$ such that $h(C) = C$. Moreover, if $D \subset S$ is another closed subset, then we denote by $\mathcal{H}_D(S) \subset \mathcal{H}(S)$ and $\mathcal{H}_D(S, C) \subset \mathcal{H}(S, C)$ the subgroups of the homeomorphisms which coincide with the identity in D. Finally, we denote by $\mathcal{M}(S)$, $\mathcal{M}(S, C)$, $\mathcal{M}_D(S)$ and $\mathcal{M}_D(S, C)$ the *mapping class groups* corresponding to the groups considered above (that is, we set $\mathcal{M} = \pi_0\, \mathcal{H}$).

For any $n \geq 1$, let $\mathcal{B}_n = \pi_1(\Gamma_n(\mathrm{Int}\, B^2), \{P_1, \dots, P_n\})$ be the *braid group* of order n of S based at $\{P_1, \dots, P_n\} \subset \mathrm{Int}\, B^2$, where $\Gamma_n(X) = (X^n - \Delta)/\Sigma_n$ denotes the configuration space of all the subsets of X with cardinality n. We recall that there exists an isomorphism $\eta : \mathcal{B}_n \to \mathcal{M}_{S^1}(B^2, \{P_1, \dots, P_n\})$, defined by setting $\eta(b)$ equal to the isotopy class of the ending homeomorphism h_1 of any isotopy $t \mapsto h_t \in \mathcal{H}_{S_1}(B^2)$ which realizes the braid b (that is, the map $t \mapsto h_t(\{P_1, \dots, P_n\})$ is a loop in $\Gamma_n(\mathrm{Int}\, B^2)$ representing b).

We use the right-handed notation for the action of braids on everything, that is $(a)b$ denotes the image of a by the action of the braid b. If a itself is a braid, then we have $(a)b = b^{-1}ab$. Moreover, we adopt the following bracketing convention:$(a)b_1 b_2 \dots b_n = (\dots((a)b_1)b_2 \dots)b_n$.

We say that a homeomorphism $h \in \mathcal{H}_{\mathrm{Bd}\, G}(G)$ is *liftable* with respect to the branched covering $p : F \to G$ iff there exists $k \in \mathcal{H}_{\mathrm{Bd}\, F}(F)$ such that $p\, k = h\, p$. We call k a *lifting* of h. Of course, for such h and k, we have $h(B_p) = B_p$ and $k(S_p) = S_p$. Moreover, the lifting k is unique if $\mathrm{Bd}\, G \neq \emptyset$, otherwise h may have more than one lifting. In any case, liftability is preserved by composition and is invariant by isotopy in $\mathcal{H}_{\mathrm{Bd}\, G}(G, B_p)$, so it makes sense to speak of the (subgroup of the) liftable isotopy classes in $\mathcal{M}_{\mathrm{Bd}_G}(G, B_p)$.

Given a simple branched covering $p : F \to B^2$, we call $b \in \mathcal{B}_n$ (the braid group based at the branch set $B_p = \{P_1, \dots, P_n\}$ of p) a *liftable braid* with respect to p iff $\eta(b) \in \mathcal{H}_{S^1}(B^2, B_p)$ is a liftable isotopy class. Moreover, we denote by $\mathcal{L}_p \subset \mathcal{B}_n$ the subgroup of the liftable braids with respect to p.

Following [4], we call *curve* for the branched covering $p : F \to B^2$ any simple arc $\alpha \subset B^2$ joining the base point $* \in S^1$ with B_p and such that $\mathrm{Int}\, \alpha \subset \mathrm{Int}\, B^2 - B_p$. Curves are considered up to ambient isotopy of B^2 modulo

$S^1 \cup B_p$. A *system of curves* is any family of curves $\alpha_1, \dots, \alpha_k \subset B^2$ such that $\alpha_i \cap \alpha_j = \{*\}$ for all $i \neq j$. A *fundamental system* is a maximal system of curves, that is a system of curves $\alpha_1, \dots, \alpha_n$ with the same cardinality of $B_p = \{P_1, \dots, P_n\}$.

For any curve α, let $\lambda_\alpha \in \pi_1(B^2 - B_p, *)$ be the homotopy class of a simple loop supported by the clockwise oriented boundary of a small regular neighborhood $N(\alpha)$ of α in B^2. In order to simplify the notation we put $\varphi_p(\alpha) = \varphi_p(\lambda_\alpha)$. If p is simple, then $\varphi_p(\alpha)$ is a transposition for any curve α. Viceversa, if $\alpha_1, \dots, \alpha_n$ is a fundamental system for p and $\varphi_p(\alpha_i)$ is a transposition for any $i = 1, \dots, n$, then p is simple.

Since $\lambda_{\alpha_1}, \dots, \lambda_{\alpha_n}$ generate $\pi_1(B^2 - B_p, *)$, for any fundamental system $\alpha_1, \dots, \alpha_n$ for p, the branched covering p is completely determined, up to equivalence, by the monodromies $\varphi_p(\alpha_1), \dots, \varphi_p(\alpha_n)$.

Following [4] again, we call *interval* for the branched covering $p : F \to B^2$ any simple arc $x \subset B^2$ joining two branch points and such that $\operatorname{Int} x \subset \operatorname{Int} B^2 - B_p$. Intervals are considered up to ambient isotopy of B^2 modulo $S^1 \cup B_p$. We call interval and we denote by the same symbol x also the counterclockwise half-twist around x and the corresponding braid in $\mathcal{B}_n$.

It immediately follows from the lifting theorem that any interval x has a liftable power. In fact, we prove in Section 2 that, if x is not liftable, then either x^2 or x^3 is liftable.

Theorem B. *For any branched covering $p : F \to B^2$, the group of the liftable braids $\mathcal{L}_p$ is finitely generated by liftable powers of intervals.*

The proof of Theorem B is given in Section 4. As a preliminary step, in Section 3 we consider the special case of $F = B^2$. In this case, we explicitly provide a set of generators, as described in the following Theorem C.

Let $p_n : B^2 \to B^2$ be the unique (up to equivalence) simple covering of order $d = n + 1$ with n branch points. For sake of simplicity, we denote by $\mathcal{L}_n \subset \mathcal{B}_n$ the group $\mathcal{L}_{p_n}$ of the liftable braids respect to p_n.

We assume the fundamental system $\alpha_1, \dots, \alpha_n$ and the numbering of the sheets of p_n be fixed, in such a way that the sequence of transpositions $\varphi(\alpha_1), \dots, \varphi(\alpha_n)$ is in the *canonical form* $(1\ 2), \dots, (d{-}1\ d)$, that is $\varphi(\alpha_i) = (i\ i{+}1)$ for each $i = 1, \dots, n$.

For each $i = 1, \dots, n-1$, we define $x_i \simeq \alpha_i \cup \alpha_{i+1}$ to be the unique interval such that $x_i \cup \alpha_i \cup \alpha_{i+1}$ is a Jordan curve whose interior does not contain any branch point. Moreover, we put $x_{i,j} = (x_i)x_{i+1} \dots x_{j-1}$, for $1 \leq i < j \leq n$.

Theorem C. *For any $n > 1$, the group $\mathcal{L}_n$ of the liftable braids with respect to the branched covering $p_n : B^2 \to B^2$ is generated by the above described braids x_i^3 and $x_{i,j}^2$, with $1 \leq i < n$ and $i + 1 < j \leq n$.*

The above theorems constitute the first results in the study of the lifting homomorphism $\lambda_p : \mathcal{L}_p \to \mathcal{M}_{\operatorname{Bd} F}(F)$, we are planning to carry out in order to find a set of normal generators of $\ker \lambda_p$ in $\mathcal{L}_p$, for any branched covering

$p : F \to B^2$. This would generalize a result obtained in [4] (see also [5]) for coverings of degree 3.

Our work is mainly aimed to get a general equivalence theorem for simple branched coverings of S^3 in terms of covering moves. In fact, the equivalence theorem for degree 3 coverings given in [9] and [10] is essentially based on the results of [4].

1. Branched coverings of B^2

Let $p : F \to B^2$ be a simple branched covering of order d. Given a fundamental system $\alpha_1, \dots, \alpha_n$ with monodromies $\varphi(\alpha_i) = (j_i\ k_i)$ for $i = 1, \dots, n$, we define the non-oriented graph $\Gamma = \Gamma_p(\alpha_1, \dots, \alpha_n)$ to have vertices $v_1, \dots, v_d$ and edges $e_1 = \{v_{j_1}, v_{k_1}\}, \dots, e_n = \{v_{j_n}, v_{k_n}\}$.

We remark that the ordering of the vertices of Γ is not relevant, since it depends on an arbitrary numbering of the sheets of p. On the contrary, the ordering of the edges contains relevant information related to the choice of the fundamental system. Therefore, we consider Γ as an *edge-ordered graph*, in such a way that p is uniquely determined by Γ up to equivalence.

Moreover, for each non-oriented edge-ordered graph Γ with d vertices and n edges, there exist a simple branched covering $p = p_\Gamma : F_\Gamma \to B^2$ of order d with n branch points and a fundamental system $\alpha_1, \dots, \alpha_n$ for p, such that $\Gamma = \Gamma_p(\alpha_1, \dots, \alpha_n)$.

Lemma 1.1. *F_Γ has the same homotopy type of Γ.*

Proof. F_Γ is homeomorphic to the topological union $D_1 \sqcup \dots \sqcup D_d$ of d discs, with a band glued between D_{j_i} and D_{k_i} for every $i = 1, \dots, n$. □

Now we want to establish when two connected edge-ordered graphs Γ and Γ', with d vertices (which can be assumed to be the same) and n edges, determine equivalent coverings p_Γ and $p_{\Gamma'}$.

For any $i = 1, \dots, n$, let O_i be the *elementary move* corresponding to the transformation (3.6) of [1], which transforms the graph Γ with edges $e_1, \dots, e_n$ in the graph Γ' with edges $e'_1, \dots, e'_n$, defined in the following way: if e_i and e_{i+1} are disjoint or they share both the endpoints, then we put $e'_i = e_{i+1}$, $e'_{i+1} = e_i$ and $e'_k = e_k$ for $k \neq i, i+1$; otherwise, if e_i and e_{i+1} share only one endpoint, say $e_i = \{a, b\}$ and $e_{i+1} = \{b, c\}$ with $a \neq c$, then we put $e'_i = \{a, c\}$, $e'_{i+1} = e_i$ and $e'_k = e_k$ for $k \neq i, i+1$. We also denote by O_i^{-1} the inverse elementary moves, defined in the obvious way.

Fixed a numbering of the vertices $v_1, \dots, v_d$ of Γ, we associate to each edge $e_i = \{v_{j_i}, v_{k_i}\}$ the transposition $\tau_i = (j_i\ k_i) \in \Sigma_d$. Then, we define $\Omega(\Gamma)$ as the conjugation class of the product $\tau_1 \cdots \tau_n$ in Σ_d. We observe that the class $\Omega(\Gamma)$ is uniquely determined by Γ (that is it does not depend on the numbering of the vertices) and is invariant with respect to elementary moves. Furthermore, it is straightforward to see that $\Omega(\Gamma) = \Omega(p_\Gamma)$.

Lemma 1.2. *Let Γ and Γ' be two connected edge-ordered graphs with d vertices and n edges. Then the coverings p_Γ and $p_{\Gamma'}$ are equivalent if and only if $\Omega(\Gamma) = \Omega(\Gamma')$.*

Proof. The 'only if' part is trivial. Viceversa, it suffices to prove that each connected edge-ordered graph Γ with d vertices and n edges can be transformed, by using elementary moves and their inverses, into a canonical form dependent only on $\Omega(\Gamma)$.

Let us denote by $c_1 \geq \cdots \geq c_m$ the cardinalities of the non-trivial orbits generated by any permutation of $\Omega(\Gamma)$ and let $l_i = c_1 + \cdots + c_i$ for each $i = 1, \ldots, m$, then we can choice as a canonical representative of $\Omega(\Gamma)$ the permutation π given by the product

$$(1\ 2) \cdots (l_1{-}1\ l_1)(l_1{+}1\ l_1{+}2) \cdots (l_2{-}1\ l_2)\ \cdots\ (l_{m-1}{+}1\ l_{m-1}{+}2) \cdots (l_m{-}1\ l_m)\,.$$

On the other hand, there exists a numbering $v_1, \ldots, v_d$ of the vertices of Γ, such that $\tau_1 \cdots \tau_n = \pi$, where $\tau_i = (j_i\ k_i)$ is the transposition associated to the edge $e_i = \{v_{j_i}, v_{k_i}\}$, for every $i = 1, \ldots, n$.

We want to transform Γ by elementary moves, leaving the numbering of the vertices fixed, in such a way that the sequence $\tau_1, \ldots, \tau_n$ becomes

$$\begin{gathered}(1\ 2), \ldots, (l_1{-}1\ l_1), (l_1\ l_1{+}1), (l_1\ l_1{+}1), (l_1{+}1\ l_1{+}2), \ldots, (l_2{-}1\ l_2), (l_2\ l_2{+}1),\\ (l_2\ l_2{+}1), \ldots, (l_{m-1}\ l_{m-1}{+}1), (l_{m-1}\ l_{m-1}{+}1), (l_{m-1}{+}1\ l_{m-1}{+}2), \ldots, (l_m{-}1\ l_m),\\ (l_m\ l_m{+}1), (l_m\ l_m{+}1), (l_m{+}1\ l_m{+}2), (l_m{+}1\ l_m{+}2), \ldots, (d{-}1\ d), (d{-}1\ d),\\ (d{-}1\ d), (d{-}1\ d), \ldots, (d{-}1\ d), (d{-}1\ d)\ ,\end{gathered}$$

where the first two rows contain the transposition sequence defining π, with additional pairs of consecutive equal transpositions inserted between disjoint cycles, and the last two rows consist of pairs of equal consecutive transpositions. Moreover: a) if π is the identity then the first two rows are empty and $l_m = m = 1$; b) if $l_m = d$ then the third row is empty; c) the fourth row contains $(n - m + l_m)/2 - d + 1$ pairs of transpositions.

We proceed by induction on n. If $n = 1$, then Γ itself has the required form, in fact the only possibility is $\tau_1 = (1\ 2)$ and $d = 2$, since Γ is connected. In the rest of the proof, we deal with the inductive step, assuming $n > 1$.

To begin with, we show how to perform moves on Γ in order to obtain a sequence $\tau_1, \ldots, \tau_n$ of the type $(j_1\ k_1), \ldots, (j_{n'}\ k_{n'}), (d{-}1\ d), \ldots, (d{-}1\ d)$, with $0 \leq n' < n$ and $j_i, k_i \neq d$ for each $i \leq n'$. First of all, by using Remark (3.7) of [1], it is easy to get a sequence having the form $(j_1\ k_1), \ldots, (j_{n'}\ k_{n'}), (j_{n'+1}\ d), \ldots, (j_n\ d)$, with $0 \leq n' < n$ and $j_i, k_i \neq d$ for each $i \leq n'$. Then, since O_i change the pair $(j_i\ d), (j_{i+1}\ d)$ with $j_i \neq j_{i+1}$ into the pair $(j_i\ j_{i+1}), (j_i\ d)$, we can limit ourselves to consider only the case $j_{n'+1} = \ldots = j_n = h$. If $h = d-1$, we have done. Otherwise, if $h \neq d-1$, we have that $n' > 0$, by connectedness, and that $n - n'$ is even, since $(d)\tau_1 \ldots \tau_n$ can only assume the value $d-1$ or d. At this point, we could get the desired form by the sequence of elementary moves $O_{n'}, \ldots, O_{n-1}, O_{n-1}, \ldots, O_{n'}$ if $(j_{n'}\ k_{n'})$ is $(h\ d{-}1)$. So, it remains to show how to obtain $e_{n'} = \{v_h, v_{d-1}\}$

without changing the edges e_i with $i > n'$. By connectedness, there exists a chain of edges $e_{i_1}, \dots, e_{i_l}$ of minimum length $l \geq 1$ connecting v_h and v_{d-1}, with $i_1, \dots, i_l \leq n'$. If $l = 1$, then $\tau_i = (h\ d{-}1)$ and we can finish by using Remark (3.7) of [1] again. If $l > 1$ and $|i_l - i_{l-1}| = 1$, then we can decrease by one the length of the chain by performing the move O_i with $i = \min(i_l, i_{l-1})$. On the other hand, if $|i_l - i_{l-1}| > 1$, then we can reduce by one the difference between i_{l-1} and i_l, by performing either $O_{i_l-1}^{-1}$ if $i_{l-1} < i_l$ or O_{i_l} if $i_{l-1} > i_l$. So we can conclude this part of the proof by induction on l and $|i_l - i_{l-1}|$.

From now on, we assume that the first n' edges of Γ do not contain v_d and that all the last $n - n'$ edges of Γ join v_{d-1} and v_d. If $n' = 0$, then we have finished ($d = 2$ and either $\pi = \mathrm{id}$ or $\pi = (1\ 2)$ depending on the parity of n).

Let us consider the case $n' > 0$. We denote by $\Gamma' \subset \Gamma$ the subgraph having vertices $v_1, \dots, v_{d-1}$ and edges $e_1, \dots, e_{n'}$. Since Γ' is connected and the permutation $\tau_1 \dots \tau_{n'}$ is of the type requested for π, we can apply the induction hypothesis, in order to transform Γ' into the canonical form, by a sequence of elementary moves and inverse of them. The same sequence of moves also transforms Γ in a canonical form, possibly except for the presence of more than two transpositions $(d{-}2\ d{-}1)$ immediately before the $n - n'$ transpositions $(d{-}1\ d)$. In fact, the canonical form for Γ contains either one transposition $(d{-}2\ d{-}1)$ if $c_m > 2$ or two of them if $c_m = 2$. Hence, to complete the proof, it suffices to change all the n'' exceeding transpositions $(d{-}2\ d{-}1)$ into $(d{-}1\ d)$. Taking into account that such transpositions are preceded by at least one more $(d{-}2\ d{-}1)$ and followed by $(d{-}1\ d)$ and that their number n'' is even, we can realize the wanted change by the sequence of elementary moves $O_{n'}, \dots, O_{n'-n''+1}, O_{n'-n''+1}, \dots, O_{n'}, O_{n'-n''}, \dots, O_{n'-1}, O_{n'-1}, \dots, O_{n'-n''}$. $\square$

At this point, we can prove the Theorem A stated in the introduction.

Proof of Theorem A. Let Γ and Γ' two edge-ordered graphs such that $p = p_\Gamma$ and $p' = p_{\Gamma'}$. By Lemma 1.1, Γ and Γ' are both connected. Moreover, we have $\Omega(\Gamma) = \Omega(p) = \Omega(p') = \Omega(\Gamma')$. Therefore, Theorem A immediately follows from Lemma 1.2. $\square$

We conclude this section by considering some elementary properties of the restrictions of a covering of B^2 over subdisks, which we will need in the following sections.

Given a simple branched covering $p : F \to B^2$ with n branch points and a system of curves $\alpha_1, \dots, \alpha_k \subset B^2$ for p, we denote by $p^{\alpha_1,\dots,\alpha_k} : F^{\alpha_1,\dots,\alpha_k} \to B^2_{\alpha_1,\dots,\alpha_k}$ the restriction of p to $F^{\alpha_1,\dots,\alpha_k} = p^{-1}(B^2_{\alpha_1,\dots,\alpha_k})$, where $B^2_{\alpha_1,\dots,\alpha_k}$ is the disk $B^2 - \mathrm{Int}\, N(\alpha_1, \dots, \alpha_k)$, being $N(\alpha_1, \dots, \alpha_k)$ a regular neighborhood of $\alpha_1 \cup \dots \cup \alpha_k$ that does not contain any branch points other than the endpoints of the curves $\alpha_1, \dots, \alpha_k$.

We remark that $p^{\alpha_1,\dots,\alpha_k}$ is a simple covering uniquely determined up to equivalence, which has the same order of p and $n - k$ branch points. Moreover, if p has c components, $p^{\alpha_1,\dots,\alpha_k}$ has c' components, with $c \leq c' \leq c + k$.

As base point for $B^2_{\alpha_1,\dots,\alpha_k}$ we can choose either the starting-point $*'$ or the ending-point $*''$ of the arc $N(\alpha_1,\dots,\alpha_k)\cap S^1$, oriented accordingly with the counterclockwise orientation of S^1. We denote by $\omega'_{\alpha_1,\dots,\alpha_k}$ (resp. $\omega''_{\alpha_1,\dots,\alpha_k}$) the simple clockwise oriented loop based at $*'$ (resp. $*''$) and supported by $\mathrm{Bd}\,B^2_{\alpha_1,\dots,\alpha_k}$. The liftings of the arc $N(\alpha_1,\dots,\alpha_k)\cap S^1$ determine bijections $p^{-1}(*)\cong p^{-1}(*')\cong p^{-1}(*'')$. By means of this bijections, any numbering of the sheets of p induces a *coherent numbering* of the sheets of $p^{\alpha_1,\dots,\alpha_k}$, depending on the choice of $*'$ or $*''$ as base point $B^2_{\alpha_1,\dots,\alpha_k}$.

The sheets of any restriction $p^{\alpha_1,\dots,\alpha_k}$ of p will be ever numbered coherently with the ones of p, whichever will be the choice of the base point for $B^2_{\alpha_1,\dots,\alpha_k}$. We will denote with the same letter φ the monodromy of $p^{\alpha_1,\dots,\alpha_k}$, with respect to this coherent numbering, as well as the monodromy of p, without any explicit reference to the choice of the base point.

Given any curve $\alpha\subset B^2$ for p such that $\alpha\cap(\alpha_1\cup\ldots\cup\alpha_k)=\{*\}$, we can assume (up to isotopy) that $\alpha\cap B^2_{\alpha_1,\dots,\alpha_k}$ is an arc. Then, we denote by α' and (resp. α'') the curve in $B^2_{\alpha_1,\dots,\alpha_k}$ obtained by sliding the initial point of $\alpha\cap B^2_{\alpha_1,\dots,\alpha_k}$ along the arc $N(\alpha_1,\dots,\alpha_k)\cap B^2_{\alpha_1,\dots,\alpha_k}$ until $*'$ (resp. $*''$) is reached. By using these notations we can write $p^{\alpha_1,\dots,\alpha_k}\cong(p^{\alpha_1,\dots,\alpha_h})^{\alpha''_{h+1},\dots,\alpha''_k}\cong(p^{\alpha_{h+1},\dots,\alpha_k})^{\alpha'_1,\dots,\alpha'_h}$ for each $h=1,\dots,k-1$.

If $\alpha_1,\dots,\alpha_n$ is a fundamental system for p, then for any $i_1<\ldots<i_k$ and $j_1<\ldots<j_{n-k}$ such that $\{i_1,\dots,i_k\}\cup\{j_1,\dots,j_{n-k}\}=\{1,\dots,n\}$, we have that $\alpha'_{j_1},\dots,\alpha'_{j_{n-k}}$ (resp. $\alpha''_{j_1},\dots,\alpha''_{j_{n-k}}$) is a fundamental system for $p^{\alpha_{i_1},\dots,\alpha_{i_k}}$ with base point $*'$ (resp. $*''$). Moreover, by putting $\tau_i=\varphi(\alpha_i)$, straightforward computations give the following equalities: $\varphi(\alpha'_j)=\tau_j$ and $\varphi(\alpha''_j)=(\tau_j)\tau_{i_1}\ldots\tau_{i_k}$, if $j<i_1$; $\varphi(\alpha'_j)=(\tau_j)\tau_{i_l}\ldots\tau_{i_1}$ and $\varphi(\alpha''_j)=(\tau_j)\tau_{i_{l+1}}\ldots\tau_{i_k}$, for $i_l<j<i_{l+1}$ with $1\le l\le k-1$; $\varphi(\alpha'_j)=(\tau_j)\tau_{i_k}\ldots\tau_{i_1}$ and $\varphi(\alpha''_j)=\tau_j$, if $j>i_k$.

Lemma 1.3. *If $p:F\to B^2$ is a simple covering and $\alpha_1,\dots,\alpha_k\subset B^2$ is a system of curves for p, then $\varphi(\omega'_{\alpha_1,\dots,\alpha_k})=\varphi(\omega)\varphi(\alpha_k)\ldots\varphi(\alpha_1)$ and $\varphi(\omega''_{\alpha_1,\dots,\alpha_k})=\varphi(\alpha_k)\ldots\varphi(\alpha_1)\varphi(\omega)$.*

Proof. Let $\alpha_{k+1},\dots,\alpha_n$ be curves such that $\alpha_1,\dots,\alpha_n$ is a fundamental system for p. Then, by the equalities above, $\varphi(\omega'_{\alpha_1,\dots,\alpha_k})=\varphi(\alpha'_{k+1})\ldots\varphi(\alpha'_n)=(\varphi(\alpha_{k+1})\ldots\varphi(\alpha_n))\varphi(\alpha_k)\ldots\varphi(\alpha_1)=\varphi(\omega)\varphi(\alpha_k)\ldots\varphi(\alpha_1)$ and $\varphi(\omega''_{\alpha_1,\dots,\alpha_k})=\varphi(\alpha''_{k+1})\ldots\varphi(\alpha''_n)=\varphi(\alpha_{k+1})\ldots\varphi(\alpha_n)=\varphi(\alpha_k)\ldots\varphi(\alpha_1)\varphi(\omega)$. □

Lemma 1.4. *A connected simple covering $p:F\to B^2$ with n branch points is equivalent to p_n if and only if one of the following conditions holds: (1) $F\cong B^2$; (2) p has order $n+1$; (3) p^α is disconnected for every curve α.*

Proof. $p\cong p_n\Rightarrow$ (1) is trivial. (1) $\Rightarrow$ (2) follows from Lemma 1.1. (2) $\Rightarrow$ (3) follows from the fact that $n-1$ transpositions cannot generate a transitive action on $\{1,\dots,n+1\}$. In order to prove the implication (3) $\Rightarrow p\cong p_n$, we consider a fundamental system $\alpha_1,\dots,\alpha_n$ for p such that the sequence of transpositions $\varphi_p(\alpha_1),\dots,\varphi_p(\alpha_1)$ has the canonical form obtained in the proof of Lemma 1.2. It is easy to see that $p \cong p_n$ iff $\varphi(\alpha_m) \neq \varphi(\alpha_{m+1})$ for each $m = 1, \dots, n-1$. On the other hand, if $\varphi(\alpha_m) = \varphi(\alpha_{m+1})$ for some m, then the restriction p^{α_m} is connected. □

2. Liftable braids and intervals

In this section we consider a simple branched covering $p : F \to B^2$ of degree d with n branch points and denote by φ its monodromy. We denote by $\mathcal{B}_n$ the braid group based at the branch set B_p of p and by $\mathcal{L}_p \subset \mathcal{B}_n$ the subgroups of the liftable braids with respect to p.

Let us begin with some elementary properties of liftable braids. The following liftability criterion in terms of action on a fundamental system will play a crucial role.

Lemma 2.1. *Let $\alpha_1, \dots, \alpha_n$ be a fundamental system for p. Then, a braid $b \in \mathcal{B}_n$ is liftable if and only if $\varphi((\alpha_i)b) = \varphi(\alpha_i)$ for every $i = 1, \dots, n$.*

Proof. Let b_* be the automorphism of $\pi_1(B^2 - B_p, *)$ induced by the restriction of b to $B^2 - B_p$. By the lifting theorem, b is liftable if and only if $\varphi\, b_* = \varphi$, since the lifting of b is the identity on $p^{-1}(*)$ and thus it induces the identity conjugation on Σ_d. Then, the statement follows from the fact that the sequence of transpositions associated to a fundamental system uniquely determines φ. □

Let $\alpha_1, \dots, \alpha_k$ and $\beta_1, \dots, \beta_k$ two systems of curves. Suppose that there exists a liftable braid $b \in \mathcal{B}_n$ such that $(\alpha_i)b = \beta_i$ for every $i = 1, \dots, k$. Since any system of curves can be completed to a fundamental system, by the previous lemma, we have $\varphi(\alpha_i) = \varphi(\beta_i)$ for every $i = 1, \dots, k$. On the other hand, we can always suppose that $(B^2_{\alpha_1,\dots,\alpha_k})b = B^2_{\beta_1,\dots,\beta_k}$, up to isotopy. Therefore, the restriction $b_| : B^2_{\alpha_1,\dots,\alpha_k} \to B^2_{\beta_1,\dots,\beta_k}$ induces an equivalence between $p^{\alpha_1,\dots,\alpha_k}$ and $p^{\beta_1,\dots,\beta_k}$, preserving the numbering of the sheets, when they are referred to the same base point $*'$ or $*''$.

Lemma 2.2. *Given two systems of curves $\alpha_1, \dots, \alpha_k$ and $\beta_1, \dots, \beta_k$ for p, there exists a liftable braid $b \in \mathcal{B}_n$ such that $(\alpha_i)b = \beta_i$ for every $i = 1, \dots, k$ if and only if the following conditions hold:* (1) *$\varphi(\alpha_i) = \varphi(\beta_i)$ for every $i = 1, \dots, k$;* (2) *there exists a bijection between the components of $p^{\alpha_1,\dots,\alpha_k}$ and $p^{\beta_1,\dots,\beta_k}$ preserving the number of branch points and the numbering of the sheets, when they are referred to the same base point.*

Proof. The 'only if' part immediately follows from the previous discussion. In order to prove the converse, it suffices to extend the given systems of curves to fundamental systems $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ such that $\varphi(\alpha_i) = \varphi(\beta_i)$ for every $i = 1, \dots, n$. In fact, in this case the braid $b \in B_n$ uniquely defined by $(\alpha_i)b = \beta_i$ for $i = 1, \dots, n$ turns out to be liftable by Lemma 2.1. The fundamental systems $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ will be constructed by induction on $m = n - k$.

If $m = 0$ there is nothing to do. So, we assume $m > 0$ and observe that in this case there exist connected components $C \subset F^{\alpha_1,\dots,\alpha_k}$ and $D \subset F^{\beta_1,\dots,\beta_k}$, such that the restrictions $p^{\alpha_1,\dots,\alpha_k}_{|C} : C \to B^2_{\alpha_1,\dots,\alpha_k}$ and $p^{\beta_1,\dots,\beta_k}_{|D} : D \to B^2_{\beta_1,\dots,\beta_k}$ are non-trivial. We assume that such components correspond each other with respect to the bijection of property (2). Then, they involve the same sheets $\{i_1, \dots, i_e\}$ and the same number $l > 0$ of branch points. Moreover, by Lemma 1.3, they have the same total monodromy with respect to the base point $*''$, that is the restriction of $\varphi(\omega''_{\alpha_1,\dots,\alpha_k}) = \varphi(\omega''_{\beta_1,\dots,\beta_k})$ to $\{i_1, \dots, i_e\}$.

By the proof of Theorem A, it is possible to construct two fundamental systems $\gamma_1, \dots, \gamma_l$ for $p^{\alpha_1,\dots,\alpha_k}_{|C}$ and $\delta_1, \dots, \delta_l$ for $p^{\beta_1,\dots,\beta_k}_{|D}$, with the same base point $*''$ and such that $\varphi(\gamma_i) = \varphi(\delta_i)$ for every $i = 1, \dots, l$.

Now we consider the systems of curves $\alpha_1, \dots, \alpha_{k+l}$ and $\beta_1, \dots, \beta_{k+l}$, extending the original ones in such a way that $\alpha''_i = \gamma_{i-k}$ and $\beta''_i = \delta_{i-k}$ for all $i = k+1, \dots, k+l$. Properties (1) and (2) still hold for these new systems of curves. Therefore, by the induction hypothesis, they can be further extended to fundamental systems as desired. □

We remark that property (2) in the statement of Lemma 2.2 trivially follows from property (1) when the restrictions $p^{\alpha_1,\dots,\alpha_k}$ and $p^{\beta_1,\dots,\beta_k}$ are connected. More generally, this fact holds also when the two restrictions have at most one non-trivial component and their trivial sheets are numbered in the same way.

In the rest of this section, we deal with intervals. Given an interval $x \subset B^2$ for the covering p, we say that x is of *type* i iff x^i is the first positive power of x which is liftable with respect to p as a braid.

By the following lemma (cf. Lemma 2.4 of [4]), each interval is either of type 1 or type 2 or type 3. Moreover, it can be easily realized that the intervals x and $(x)b$ are of the same type for each liftable braid $b \in \mathcal{L}_p$.

Lemma 2.3. *Let x be an interval for p and α be a curve for p meeting x only at one of its endpoints. Then: x is of type 1 if and only if $\varphi(\alpha) = \varphi((\alpha)x)$; x is of type 2 if and only if $\varphi(\alpha)$ and $\varphi((\alpha)x)$ are disjoint transpositions; x is of type 3 if and only if $\varphi(\alpha)$ and $\varphi((\alpha)x)$ are different and not disjoint.*

Proof. Given x and α as in the statement, let $\alpha_1, \dots, \alpha_n$ be a fundamental system such that $\alpha_1 = \alpha$, $\alpha_2 = (\alpha)x$ and $(\alpha_i)x = \alpha_i$ for $i = 3, \dots, n$. By Lemma 2.1, x is liftable iff it preserves all the monodromies of such fundamental system, that is iff $\varphi(\alpha_1) = \varphi(\alpha_2)$. The other two cases can be achieved by similar applications of Lemma 2.1 to the intervals x^2 and to x^3, taking into account that $\varphi((\alpha)x^2) = \varphi((\alpha)x)\varphi(\alpha)\varphi((\alpha)x)$ and $\varphi((\alpha)x^3) = \varphi((\alpha)x^2)\varphi((\alpha)x)\varphi((\alpha)x^2) = \varphi((\alpha)x)\varphi(\alpha)\varphi((\alpha)x)\varphi(\alpha)\varphi((\alpha)x)$. □

We denote by $\mathcal{I}_p \subset \mathcal{L}_p$ the subgroup generated by all the liftable powers of intervals, that is by the intervals of type 1, the second power of the intervals of type 2 and the third power of the intervals of type 3. Of course, Theorem B says that $\mathcal{I}_p = \mathcal{L}_p$. Nevertheless, it is temporarily convenient to keep different notations for the two groups.

Fixed a fundamental system $\alpha_1, \dots, \alpha_n$ for p, we call *index* of a curve or an interval (with respect to $\alpha_1, \dots, \alpha_n$) the minimum number (up to isotopy) of the intersections with $\alpha_1 \cup \dots \cup \alpha_n$, not including the endpoints.

Moreover, depending on the fundamental system $\alpha_1, \dots, \alpha_n$, we give the following definitions: $x_i \simeq \alpha_i \cup \alpha_{i+1}$ is the unique interval such that $x_i \cup \alpha_i \cup \alpha_{i+1}$ is a Jordan curve whose interior does not contain any branch point, for $i = 1, \dots, n-1$; $x_{i,j} = (x_i)x_{i+1} \dots x_{j-1}$, for $1 \le i < j \le n$; $\widehat{x}_{i,j} = (x_i)x_{i+1}^{-1} \dots x_{j-1}^{-1}$, for $1 \le i < j \le n$; in addition, as a notational convenience, we put $x_{i,j} = x_{j,i}$ and $\widehat{x}_{i,j} = \widehat{x}_{j,i}$, for $1 \le j < i \le n$. In particular, we have $x_i = x_{i,i+1} = \widehat{x}_{i,i+1}$.

We remark that the braids $x_1, \dots, x_{n-1}$ are the usual standard generators of the braid group $\mathcal{B}_n$; similarly, the braids $x_{i,j}^2$ (as well as the braids $\widehat{x}_{i,j}^2$) with $1 \le i < j \le n$ are standard generators of the pure braids group $\mathcal{P}_n \subset \mathcal{B}_n$.

We conclude this section by considering all the intervals and all the curves of indices 0 and 1 with respect to the fixed fundamental system $\alpha_1, \dots, \alpha_n$.

The intervals of index 0 are the $\widehat{x}_{i,j}$'s. The curves of index 0 are the curves $\alpha_{i,j}$ with $1 \le i, j \le n$, defined in the following way: $\alpha_{i,i} = \alpha_i$, $\alpha_{i,j} = (\alpha_i)\widehat{x}_{i,j}^{-1}$ if $i < j$, $\alpha_{i,j} = (\alpha_i)\widehat{x}_{i,j}$ if $j < i$. Such intervals and curves are related by the following equalities:

$$(\alpha_{i,j})\widehat{x}_{j,k}^{-1} = \begin{cases} \alpha_{i,k} & \text{if } \ i \le j < k \ \text{ or } \ j < k \le i\,, \\ \alpha_{i-1,k} & \text{if } \ k < i \le j\,; \end{cases}$$

$$(\alpha_{i,j})\widehat{x}_{j,k} = \begin{cases} \alpha_{i,k} & \text{if } \ i \le k < j \ \text{ or } \ k < j \le i\,, \\ \alpha_{i+1,k} & \text{if } \ j \le i < k\,. \end{cases}$$

The intervals of index 1 are the intervals $\widehat{x}_{i,j,k}$ with $1 \le i, j, k \le n$ such that $i < k$ and $i \ne j \ne k$, given by: $\widehat{x}_{i,j,k} = (\widehat{x}_{i,j})\widehat{x}_{j,k}$ if $i < j < k$ and $\widehat{x}_{i,j,k} = (\widehat{x}_{i,j})\widehat{x}_{j,k}^{-1}$ if $i < k < j$ or $j < i < k$. As a notational convenience, we also set $\widehat{x}_{i,j,k} = \widehat{x}_{k,j,i}$ if $i > k$ and $\widehat{x}_{i,i,j} = \widehat{x}_{i,j,j} = \widehat{x}_{i,j}$ for every $i \ne j$.

Finally, the curves of index 1 are the curves $\alpha_{i,j,k}$ with $1 \le i, j, k \le n$ such that $i \ne j \ne k$, defined as follows:

$$\alpha_{i,j,k} = \begin{cases} (\alpha_{i,j})\widehat{x}_{j,k} & \text{if } \ i < j < k \ \text{ or } \ j < k \le i \ \text{ or } \ k < i < j\,, \\ (\alpha_{i,j})\widehat{x}_{j,k}^{-1} & \text{if } \ k < j < i \ \text{ or } \ j < i < k \ \text{ or } \ i \le k < j\,. \end{cases}$$

3. Liftable braids with respect to $p : B^2 \to B^2$

By the results of Section 1, for every $n \ge 1$ there exists a unique (up to equivalence) simple branched covering $p_n : B^2 \to B^2$ of order $d = n + 1$ with n branch points. Moreover, the p_n's represent (up to equivalence) all the coverings of B^2 onto itself.

We assume the base point $* \in S^1$, the branch points $P_1, \dots, P_n \in \operatorname{Int} B^2$, the fundamental system $\alpha_1, \dots, \alpha_n$ and the numbering of the sheets of p_n fixed in such a way that: (1) α_i joins $*$ to P_i for every $i = 1, \dots, n$; (2) the monodromy sequence $\varphi(\alpha_1), \dots, \varphi(\alpha_n)$ is in the canonical form $(1\ 2), \dots, (d{-}1\ d)$ given in the proof of Theorem A, namely $\varphi(\alpha_i) = (i\ i{+}1)$ for every $i = 1, \dots, n$.

In this section, all the curves $\alpha_{i,j}$ and $\alpha_{i,j,k}$ and all the intervals x_i, $x_{i,j}$, $x_{i,j,k}$, $\widehat{x}_{i,j}$, $\widehat{x}_{i,j,k}$, as well as all the indexes of curves and intervals, except where expressly indicated, are referred to the fundamental system $\alpha_1, \dots, \alpha_n$.

In order to prove Theorem C, let us begin with some preliminary results about curves. We recall that $\mathcal{L}_n \subset \mathcal{B}_n$ denotes the subgroup of the liftable braids with respect to p_n.

By direct computation we get the following monodromies:

$$\varphi(\alpha_{i,j}) = \begin{cases} (i\ j{+}1) & \text{if } \ i \le j\,, \\ (i{+}1\ j) & \text{if } \ j \le i\,; \end{cases}$$

$$\varphi(\alpha_{i,j,k}) = \begin{cases} (j{+}1\ k{+}1) & \text{if } \ i < j < k \ \text{ or } \ i \le k < j\,, \\ (j\ k) & \text{if } \ k < j < i \ \text{ or } \ j < k \le i\,, \\ (j{+}1\ k) & \text{if } \ k \le i < j\,, \\ (j\ k{+}1) & \text{if } \ j < i \le k\,. \end{cases}$$

Assuming $n > 1$, we say that a curve α for p_n is *regular* if p_n^α is equivalent to $p_{n-1} \sqcup \mathrm{id}_{B^2}$. We observe that, if α is a regular curve then $(\alpha)b$ is a regular curve for any liftable braid $b \in \mathcal{L}_n$.

Lemma 3.1. *The curve α_j is regular for every $j = 1, \dots, n$. Moreover, the equivalence between p_{n-1} and the non-trivial component of $p_n^{\alpha_j}$ is induced by a homeomorphism $h_j : B^2 \to B^2_{\alpha_j}$ such that: $h_j(\alpha_i) = \alpha'_i$ for $1 \le i < j$; $h_j(\alpha_i) = \alpha'_{i+1}$ for $j \le i \le n-1$; $h_j(x_i) = x_i$ for $1 \le i < j-1$; $h_j(x_{j-1}) = (x_{j-1})x_j^{-1}$; $h_j(x_i) = x_{i+1}$ for $j - 1 < i \le n-2$.*

Proof. The fundamental system $\alpha'_1, \dots, \alpha'_{j-1}, \alpha'_{j+1}, \dots, \alpha'_n$ for $p_n^{\alpha_j}$ has monodromy sequence $(1\ 2), \dots, (j{-}2\ j{-}1), (j{-}1\ j{+}1), (j{+}1\ j{+}2), \dots, (n{-}1\ n)$. Let $h_j : B^2 \to B^2_{\alpha_j}$ be the homeomorphism uniquely determined (up to isotopy) by $h_j(\alpha_i) = \alpha'_i$ for $1 \le i < j$ and $h_j(\alpha_i) = \alpha'_{i+1}$ for $j \le i \le n-1$. By the Lifting theorem, h_j can be lifted to give an equivalence between $p_{n-1} \sqcup \mathrm{id}_{B^2}$ and $p_n^{\alpha_j}$. Hence, h_j induces an equivalence between p_{n-1} and the non-trivial component of $p_n^{\alpha_j}$. A straightforward computation of the intervals $h_j(x_i)$ completes the proof. □

Lemma 3.2. *The curves $\alpha_{1,n}$ e $\alpha_{n,1}$ are regular. Moreover, we have that: the equivalence between p_{n-1} and the non-trivial component of $p_n^{\alpha_{1,n}}$ is induced by a homeomorphism $h_{1,n} : B^2 \to B^2_{\alpha_{1,n}}$ such that $h_{1,n}(\alpha_i) = \alpha''_i$ for $1 \le i \le n-1$ and $h_{1,n}(x_i) = x_i$ for $1 \le i \le n-2$; the equivalence between p_{n-1} and the non-trivial component of $p_n^{\alpha_{n,1}}$ is induced by a homeomorphism $h_{n,1} : B^2 \to B^2_{\alpha_{n,1}}$ such that $h_{n,1}(\alpha_i) = \alpha'_{i+1}$ for $1 \le i \le n-1$ and $h_{n,1}(x_i) = x_{i+1}$ for $1 \le i \le n-2$.*

Proof. Similar to the previous one, except that we consider the fundamental system $\alpha''_1, \ldots, \alpha''_{n-1}$ instead of $\alpha'_1, \ldots, \alpha'_{n-1}$ for the covering $p_n^{\alpha_{1,n}}$. $\square$

Lemma 3.3. *The only regular curves of index* 0 *are* $\alpha_1, \ldots, \alpha_n$, $\alpha_{1,n}$ *and* $\alpha_{n,1}$. *Among these, only* $\alpha_{1,n}$ *and* $\alpha_{n,1}$ *are* $\mathcal{L}_n$*-equivalent to each other.*

Proof. Lemmas 3.1 and 3.2 say that the curves $\alpha_1, \ldots, \alpha_n, \alpha_{1,n}$ and $\alpha_{n,1}$ are regular. In the previous section, we observed that any other curve of index 0 have to be an $\alpha_{i,j}$ with $j \neq i$ and $(1, n) \neq (i, j) \neq (n, 1)$. If $j > i$, then the curves $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i,j}, \alpha_i, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_n$ constitute a fundamental system for p_n with monodromy sequence

$$(1\,2), \ldots, (i{-}1\; i), (i\; j{+}1), (i\; i{+}1), \ldots, (j{-}1\; j), (j{+}1\; j{+}2), \ldots, (n\; n{+}1).$$

If $j < i$ then the curves $\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_i, \alpha_{i,j}, \alpha_{i+1}, \ldots, \alpha_n$ constitute a fundamental system for p_n with sequence of monodromies

$$(1\,2), \ldots, (j{-}1\; j), (j{+}1\; j{+}2), \ldots, (i\; i{+}1), (j\; i{+}1), (i{+}1\; i{+}2), \ldots, (n\; n{+}1).$$

In both cases, none of the curves $\alpha_{i,j}$ is regular, as can be immediately proved by using Lemma 1.4.

For the second part of the lemma, we observe that the monodromies of the curves taken into account are distinct from each other, with the only exception of $\varphi(\alpha_{1,n}) = \varphi(\alpha_{n,1}) = (1\; n{+}1)$. On the other hand, since $\alpha_{n,1} = (\alpha_{1,n})b$, with $b = (x_{n-1} \ldots x_1)^{n+1} \in \mathcal{L}_n$, we have that $\alpha_{1,n}$ and $\alpha_{n,1}$ are $\mathcal{L}_n$-equivalent. $\square$

Lemma 3.4. *Any fundamental system* $\beta_1, \ldots, \beta_n$ *for* p_n *with* $n > 1$, *contains at least two regular curves* β_{i_1} *e* β_{i_2}.

Proof. Let $\Gamma = \Gamma_{p_n}(\beta_1, \ldots, \beta_n)$ be the graph associated to $\beta_1, \ldots, \beta_n$. Moreover, for any $i = 1, \ldots, n$, let $\Gamma_i = \Gamma_{p_n^{\beta_i}}(\beta'_1, \ldots, \beta'_{i-1}, \beta'_{i+1}, \ldots, \beta'_n)$ be the graph associated to the fundamental system $\beta'_1, \ldots, \beta'_{i-1}, \beta'_{i+1}, \ldots, \beta'_n$ for $p_n^{\beta_i}$. By Lemma 1.1, Γ is a tree. On the other hand, it follows from Lemma 1.4 that all the Γ_i's have two connected components and that β_i is regular if and only if one component of Γ_i consists of a single vertex. Then, it is enough to prove that there exist two graph Γ_{i_1} and Γ_{i_2} with that property.

The graph Γ_i can obtained from Γ, by removing the edge e_i and replacing the edge $e_l = \{v_{j_l}, v_{k_l}\}$ with the new edge $e_{l-1} = \{v_{\varphi(\beta_i)(j_l)}, v_{\varphi(\beta_i)(k_l)}\}$, for every $l > i$. We remark that the edges $e_1, \ldots, e_{i-1}$, as well as all the e_l's not meeting e_i, are left unaltered.

Now let Γ' be the full subgraph of Γ generated by all the vertices of valence greater than 1. It is not difficult to see that Γ collapses to Γ' (remember that $n > 1$). Then, also Γ' is a non-empty tree.

If Γ' reduces to a single vertex, this vertex is contained in all the edges $e_1, \ldots, e_n$ of Γ. In this case, we have that Γ_1 and Γ_n have the required property. Otherwise, Γ' must contain al least two different valence one vertices w_1 and w_2. From these vertices come out two different edges e_{i_1} and e_{i_2} of $\Gamma - \Gamma'$, such that the graphs Γ_{i_1} e Γ_{i_2} have the required property.

Let us see how to determine i_1 (in the same way could be determined i_2). Let e_{l_1} be the only edge of Γ' containing w_1. Since the valence of w_1 in Γ is greater than one, there is least one edge of $\Gamma - \Gamma'$ containing w_1. Then, we can set i_1 equal to the maximum among the indices of such edges. $\square$

We continue by considering some properties of the intervals. First of all, we observe that all the intervals x_i are of type 3 with respect to p_n, while all the intervals $x_{i,j}$ with $j > i+1$ are of type 2.

Lemma 3.5. *All the index* 0 *intervals are of type* 3 *with respect to* p_n*.*

Proof. We recall that the index 0 intervals are the $\widehat{x}_{i,j}$'s with $i < j$. Such intervals are of type 3 by Lemma 2.3, since the curve α_i meets $\widehat{x}_{i,j}$ only at its endpoint, $\varphi(\alpha_i) = (i\ i{+}1)$ and $\varphi((\alpha_i)\widehat{x}_{i,j}) = \varphi(\alpha_{i+1,j}) = (i{+}1\ j{+}1)$. $\square$

Lemma 3.6. *All the index* 1 *intervals are of type* 2 *with respect to* p_n*.*

Proof. We recall that the index 1 intervals are the $\widehat{x}_{i,j,k}$'s with $i < k$ and $i \neq j \neq k$. The curve α_i meets $\widehat{x}_{i,j,k}$ only at its endpoint and we have that $(\alpha_i)\widehat{x}_{i,j,k}$ coincides with $\alpha_{i+1,j,k}$ if $i < j < k$ or $i < j < k$ and with $\alpha_{i,j,k}$ if $j < i < k$. In any case, the transpositions $\varphi(\alpha_i)$ and $\varphi((\alpha_i)\widehat{x}_{i,j,k})$ are disjoint. Then, $\widehat{x}_{i,j,k}$ is of type 2 by Lemma 2.3. $\square$

Lemma 3.7. *There are no intervals of type* 1 *with respect to* p_n*.*

Proof. Given any interval x and any curve α which meets x only at its endpoint, let $\beta_1, \dots, \beta_n$ be any fundamental system such that $\beta_1 = \alpha$ and $\beta_2 = (\alpha)x$. If x were of type 1, $\varphi(\beta_1)$ would coincide with $\varphi(\beta_2)$, in contradiction with Lemma 1.1 and Lemma 1.4. $\square$

For sake of simplicity, we denote by $\mathcal{I}_n \subset \mathcal{L}_n$ the group $\mathcal{I}_{p_n}$ generated by the liftable powers of intervals. The braids x_i^3 and $x_{i,j}^2$ with $1 \leq i < n$ and $i+1 < j \leq n$ belong to $\mathcal{I}_n$. In fact, we will see that they generate $\mathcal{I}_n$.

Lemma 3.8. *If α is a curve whose interior meets each one of the curves $\alpha_1, \dots, \alpha_n$ in at most one point, then α is $\mathcal{I}_n$-equivalent to a curve of index* 0*.*

Proof. We proceed by induction on the index of α, assuming that α minimizes the number of intersection points with $\alpha_1 \cup \ldots \cup \alpha_n$ in its isotopy class.

We start with the index 1 case. In this case, we have the curves $\alpha = \alpha_{i,j,k}$, with $1 \leq i, j, k \leq n$ such that $i \neq j \neq k$, defined in Section 2. If $i = k$, it suffices to observe that $\alpha_{i,j,i}$ is $\mathcal{I}_n$-equivalent to the index 0 curve $(\alpha_{i,j,i})\widehat{x}_{i,j}^{\pm 3} = \alpha_{i\pm 1,j}$, where $\pm$ is the sign of $j - i$, being $\widehat{x}_{i,j}$ of type 3 by Lemma 3.5. If $i \neq k$, then $\alpha_{i,j,k}$ is $\mathcal{I}_n$-equivalent to $(\alpha_{i,j,k})\widehat{x}_{i,j,k}^{\pm 2}$, where $\pm$ is the sign of $j - i$, being $\widehat{x}_{i,j,k}$ of type 2 by Lemma 3.6. The curve $(\alpha_{i,j,k})\widehat{x}_{i,j,k}^{\pm 2}$ has index 0 if $|i - j| = 1$, while it coincides with the curve $\alpha_{i\pm 1,j,k}$, if $|i - j| > 1$. So, we can conclude the case of the $\alpha_{i,j,k}$'s with $i \neq k$, by induction on $|i - j| \geq 1$.

Now we suppose that α has index > 1. Let P_k be the endpoint of α and let $Q_i \in \alpha \cap \alpha_i$ and $Q_j \in \alpha \cap \alpha_j$ be respectively the last but one and the last point in which the interior of α (oriented from $*$ to P_k) meets the curves $\alpha_1, \dots, \alpha_n$. We consider the following arcs: $t_i \subset \alpha_i$ with endpoints Q_i and P_i,

$t_j \subset \alpha_j$ with endpoints Q_j and P_j, $s_i \subset \alpha$ with endpoints Q_i and P_k, $s_j \subset \alpha$ with endpoints Q_j and P_k. By hypothesis we have $i \neq j$. Moreover, we can assume $j \neq k$, otherwise we could remove the intersection Q_j up to isotopy.

If $i = k$, the interval $x = t_j \cup s_j$ has index 0. Then, by Lemma 3.5, α is $\mathcal{I}_n$-equivalent to the curve $(\alpha)x^{\pm 3}$, with sign $-$ if t_j is on the left of α and sign $+$ if t_j is on the right of α. The curve $(\alpha)x^{\pm 3}$ has index less than α (the intersections Q_i and Q_j disappear) and it is $\mathcal{I}_n$-equivalent to a curve of index 0 by the induction hypothesis.

If $i \neq k$, the interval $x = t_i \cup s_i$ has index 1. Then, by Lemma 3.6, α is $\mathcal{I}_n$-equivalent to the curve $(\alpha)x^{\pm 2}$, with sign $-$ if t_i is on the left of α and sign $+$ if t_i is on the right of α. The curve $(\alpha)x^{\pm 2}$ has index less than α (the intersection Q_i disappears) and it is $\mathcal{I}_n$-equivalent to a curve of index 0 by the induction hypothesis. □

Lemma 3.9. *Every curve α is $\mathcal{I}_n$-equivalent to a curve of index 0.*

Proof. We proceed by induction on n. For $n = 1$ there is nothing to prove. So, let us suppose $n > 1$. First of all, we consider the special case in which $\alpha \cap \alpha_j = \{*\}$ for some $j = 1, \dots, n$. By Lemma 3.1 and by the induction hypothesis, it exists a braid $b \in \mathcal{I}_{p_n^{\alpha_j}}$ such that the curve $(\alpha')b$ has index 0 with respect to the fundamental system $\alpha'_1, \dots, \alpha'_{j-1}, \alpha'_{j+1}, \dots, \alpha'_n$ for $p_n^{\alpha_j}$. The braid b can also be considered as a braid in $\mathcal{I}_n$ and it is easy to verify that the curve $(\alpha)b$ satisfies Lemma 3.8. Then α is $\mathcal{I}_n$-equivalent to a curve of index 0. By Lemma 3.2, also the cases $\alpha \cap \alpha_{1,n} = \{*\}$ and $\alpha \cap \alpha_{n,1} = \{*\}$, with the braid b respectively in $\mathcal{I}_{p_n^{\alpha_{1,n}}}$ and in $\mathcal{I}_{p_n^{\alpha_{n,1}}}$ can be treated in an analogous way.

Now we carry on the proof by induction on the index of α, assuming that α meets every α_j in some point other than $*$. For every $j = 1, \dots, n$, we denote by Q_j the point of $\alpha \cap \alpha_j$ nearest to P_j along α_j, and by β_j the curve obtained following α from $*$ to Q_j and then α_j from Q_j to P_j. If P_k is the endpoint of α, then $\beta_k = \alpha$ and all the curves β_j with $j \neq k$ have index less than α. Since the curves $\beta_1, \dots, \beta_n$, suitably renumbered, constitute a fundamental system, Lemma 3.4 ensures the existence of $l \neq k$ such that β_l is regular. By the induction hypothesis, there exists $b \in \mathcal{I}_n$ such that $(\beta_l)b$ has index 0. Then $(\beta_l)b$ coincides either with some α_j or with $\alpha_{1,n}$ or with $\alpha_{n,1}$. Hence, $(\alpha)b$ is $\mathcal{I}_n$-equivalent to a curve of index 0, being included in the cases examined at the beginning of the proof. It follows that α is as well $\mathcal{I}_n$-equivalent to a curve of index 0. □

Lemma 3.10. *$\mathcal{L}_n$ is generated by liftable powers of intervals.*

Proof. We proceed by induction on n. If $n = 1$ there is nothing to prove. If $n > 1$ and $b \in \mathcal{L}_n$, then Lemmas 3.9 and 3.3, give us a braid $c \in \mathcal{I}_n$ such that $(\alpha_n)bc = \alpha_n$, in such a way that bc can be considered as a braid in $\mathcal{L}_{p_n^{\alpha_n}}$. By the regularity of α_n and by induction hypothesis, we have $bc \in \mathcal{I}_{p_n^{\alpha_n}} \subset \mathcal{I}_n$ and therefore $b \in \mathcal{I}_n$. □

Now, let $\mathcal{J}_n \subset \mathcal{L}_n$ denote the subgroup generated by the braids x_i^3 and $x_{i,j}^2$ with $1 \leq i < n$ and $i + 1 < j \leq n$. We want to prove that actually $\mathcal{J}_n = \mathcal{L}_n$, that is our Theorem C.

To get this goal, observe that in the proof of Lemma 3.10 we do not use the liftable powers of all the intervals, but only of some particular intervals. Therefore, it is enough to show that each one of these particular intervals is $\mathcal{J}_n$-equivalent to some x_i or $x_{i,j}$.

Lemma 3.11. *Every interval $x = (x_i)x_{i+1}^{e_{i+1}} \dots x_{j-1}^{e_{j-1}}$, with $e_{i+1}, \dots, e_{j-1} = \pm 1$ and $1 \leq i < j \leq n$, is $\mathcal{J}_n$-equivalent to some $x_{h,k}$, so all the liftable powers of x belong to $\mathcal{J}_n$.*

Proof. By induction on the number of negative e_l's. If all the e_l's are positive, then $x = x_{i,j}$. Otherwise, let $m \geq i+1$ be the minimum integer such that $e_m = -1$. If $m = i+1$, then $x = (x_i)x_{i+1}^{-1}x_{i+2}^{e_{i+2}} \dots x_{j-1}^{e_{j-1}} = (y)z^2x_i^3$ with $y = (x_{i+1})x_{i+2}^{e_{i+2}} \dots x_{j-1}^{e_{j-1}}$ and $z = (x_i)x_{i+1}x_{i+2}^{e_{i+2}} \dots x_{j-1}^{e_{j-1}}$. Since y and z are $\mathcal{J}_n$-equivalent to some $x_{h,k}$ by the induction hypothesis and z is of type 2 (so $z^2 \in \mathcal{J}_n$), we have that also x is $\mathcal{J}_n$-equivalent to some $x_{h,k}$. If $m > i+1$, then $x = (x_i)x_{i+1} \dots x_{m-1}x_m^{-1}x_{m+1}^{e_{m+1}} \dots x_{j-1}^{e_{j-1}} = (t)x_{i,m}^2$ with $t = (x_i)x_{i+1} \dots x_{m-1}x_m x_{m+1}^{e_{m+1}} \dots x_{j-1}^{e_{j-1}}$. Since t is $\mathcal{J}_n$-equivalent to some $x_{h,k}$ by the induction hypothesis, also x is $\mathcal{J}_n$-equivalent to some $x_{h,k}$. $\square$

Lemma 3.12. *Every interval x of index ≤ 1 is $\mathcal{J}_n$-equivalent to some $x_{h,k}$, so all the liftable powers of x belong to $\mathcal{J}_n$.*

Proof. The intervals of index 0, that is the $\widehat{x}_{i,j}$'s have been already considered in Lemma 3.11. The same also holds for the intervals of index 1 of type $\widehat{x}_{i,j,k}$ with $i < j < k$, in fact for these intervals we have $\widehat{x}_{i,j,k} = (x_i)x_{i+1}^{-1} \dots x_{j-1}^{-1}x_j x_{j+1}^{-1} \dots x_{k-1}^{-1}$.

It remains only to deal with the intervals $\widehat{x}_{i,j,k} = (\widehat{x}_{i,j})\widehat{x}_{j,k}^{-1}$ such that either $i < k < j$ or $j < i < k$. In the first case we have that $\widehat{x}_{i,j,k}$ is $\mathcal{J}_n$-equivalent to the interval $(\widehat{x}_{i,j,k})\widehat{x}_{j,k}^3 = \widehat{x}_{i,k,j}$. In the second case we have that $\widehat{x}_{i,j,k}$ is $\mathcal{J}_n$-equivalent to the interval $(\widehat{x}_{i,j,k})\widehat{x}_{i,j}^{-3} = \widehat{x}_{j,i,k}$. Hence, in both the cases $\widehat{x}_{i,j,k}$ is $\mathcal{J}_n$-equivalent to an interval having the form considered above. $\square$

Proof of Theorem C. We proceed by induction on n. For $n = 1$ there is nothing to prove. So, let us suppose $n > 1$. In the proof of Lemma 3.8, the $\mathcal{I}_n$-equivalence desired is obtained by using liftable powers of intervals of index ≤ 1, which belong in $\mathcal{J}_n$ by Lemma 3.12. On the other hand, in proofs of Lemmas 3.9 and 3.10, we use liftable powers of intervals in $\mathcal{I}_{p_n^{\alpha_j}}$, $\mathcal{I}_{p_n^{\alpha_{1,n}}}$ and $\mathcal{I}_{p_n^{\alpha_{n,1}}}$. By the induction hypothesis, these groups are generated by braids of the form y_i^3 and $y_{h,k}^2$ with $y_i = h(x_i)$ and $y_{h,k} = h(x_{h,k})$, where h denotes one of the homeomorphism h_j, $h_{1,n}$ and $h_{n,1}$ given by Lemmas 3.1 and 3.2. It is not difficult to see that the intervals y_i and $y_{h,k}$ are among the ones considered in Lemma 3.11, so their liftable powers belong to $\mathcal{J}_n$.

Then, we can replace the group $\mathcal{I}_n$ with the group $\mathcal{J}_n$ in Lemmas 3.8 and 3.9 as well as in the proof of Lemma 3.10, in order to get $\mathcal{L}_n = \mathcal{J}_n$. $\square$

4. Liftable braids with respect to $p : F \to B^2$

All this section is devoted to prove Theorem B. Here, we consider an arbitrary connected simple branched covering $p : F \to B^2$ of order d with n branch points. As in the previous section, we assume the base point $* \in S^1$, the branch points $P_1, \ldots, P_n \in \operatorname{Int} B^2$, the fundamental system $\alpha_1, \ldots, \alpha_n$ and the numbering of the sheets of p fixed in such a way that: (1) α_i joins $*$ to P_i for every $i = 1, \ldots, n$; (2) the monodromy sequence $\varphi(\alpha_1), \ldots, \varphi(\alpha_n)$ is in the canonical form given in the proof of Theorem A.

Lemma 4.1. *Let β be a curve such that p^β is connected and let $\beta_1, \ldots, \beta_n$ be a fundamental system for p. Then β is $\mathcal{I}_p$-equivalent to a curve γ such that $\gamma \cap \beta_i = \{*\}$ for some $i = 1, \ldots, n$.*

Proof. Let γ be a curve of minimum index with respect to the fundamental system $\beta_1, \ldots, \beta_n$ among all the curves $\mathcal{I}_p$-equivalent to β. Let us also assume that γ minimizes the number of intersection points with $\beta_1 \cup \ldots \cup \beta_n$ in its isotopy class. We claim that there exists an integer $i = 1, \ldots, n$ such that $\gamma \cap \beta_i = \{*\}$.

Suppose, by the contrary, that γ meets any β_i in some point other than $*$. For each $i = 1, \ldots, n$, we denote by Q_i the last point of $\gamma \cap \beta_i$ along β_i (starting from $*$) and with γ_i the curve obtained following γ until Q_i and then β_i until its endpoint. Up to isotopy, we can suppose $\gamma_i \cap \gamma_j = \{*\}$ for all $i \neq j$. If the endpoint of γ coincides with the endpoint of β_k, then $\gamma_k = \gamma$ and any curve γ_i with $i \neq k$ has index less than γ. We denote by $\sigma_i = \varphi(\gamma_i)$ the monodromy of γ_i. In particular, let $\sigma_k = (a\ b)$ be the monodromy of γ.

Let us consider the intervals $y_{i,j} \simeq \gamma_i \cup \gamma_j$ for $i \neq j$ and $1 \leq i, j \leq n$. We observe that all the $y_{i,k}$'s are of type 3, that is any transposition σ_i with $i \neq k$ is distinct but not disjoint from $(a\ b)$. Indeed, if $y_{i,k}$ were of type 1 or 2 then γ would be $\mathcal{I}_p$-equivalent to the curve $(\gamma) y_{i,k}^{\pm 2}$, with $-$ or $+$ depending on whether γ_i is on the left or on the right of γ, which has index less than γ.

On the other hand, if γ_i and γ_j, with $i, j \neq k$, are on the same side with respect to γ, then $\{\sigma_i, \sigma_j\} \neq \{(a\ c), (b\ c)\}$. Indeed, assuming that Q_i precedes Q_j along γ (starting from $*$), the equality $\{\sigma_i, \sigma_j\} = \{(a\ c), (b\ c)\}$ would imply the liftability of the interval $x = (y_{j,k}) y_{i,j}^{\pm 2}$, with $-$ or $+$ depending on the fact that γ_i and γ_j are on the left or on the right of γ. Therefore, γ would be $\mathcal{I}_p$-equivalent to the curve $\delta = (\gamma) x^{\pm 1}$, with the same choice for the sign, which has index less than γ.

Analogously, if γ_i and γ_j, with $i, j \neq k$, are on opposite sides with respect to γ, then $\sigma_i \neq \sigma_j$. Indeed, assuming as above that Q_i precedes Q_j along γ (starting from $*$), the equality $\sigma_i = \sigma_j$ would imply the liftability of $y_{i,j}$. Therefore, γ would be $\mathcal{I}_p$-equivalent to the curve $\delta = (\gamma) y_{i,j}^{\pm 1}$, with $-$ or $+$ depending on the fact that γ_i is on the left or on the right of γ, which has index less than γ.

Hence, by renumbering the γ_i's in clockwise order, we get a new fundamental system for p, whose monodromy sequence has the form

$$(c_1\ d_1), \dots, (c_{h-1}\ d_{h-1}), (a\ b), (c_{h+1}\ d_{h+1}), \dots, (c_n\ d_n)$$

and satisfies the following properties: $c_i \notin \{a, b\}$ and $d_i \in \{a, b\}$ for any $i \neq h$; if $i, j < h$ or $i, j > h$ then $c_i = c_j \Rightarrow d_i = d_j$; if $i < h < j$ then $c_i = c_j \Rightarrow d_i \neq d_j$. Then, by putting $C_a^- = \{c_i \mid d_i = a \wedge i < h\}$, $C_a^+ = \{c_i \mid d_i = a \wedge i > h\}$, $C_b^- = \{c_i \mid d_i = b \wedge i < h\}$ and $C_b^+ = \{c_i \mid d_i = b \wedge i > h\}$, we have $C_a^- \cap C_b^- = C_a^+ \cap C_b^+ = C_a^- \cap C_a^+ = C_b^- \cap C_b^+ = \emptyset$.

Now, the fundamental system $\gamma'_1, \dots, \gamma'_{h-1}, \gamma'_{h+1}, \dots, \gamma'_n$ for the covering p^γ has monodromy sequence $(c_1\ d_1), \dots, (c_{h-1}\ d_{h-1}), (c_{h+1}\ \bar{d}_{h+1}), \dots, (c_n\ \bar{d}_n)$, where $\bar{d}_i = a$ if $d_i = b$ and $\bar{d}_i = b$ if $d_i = a$. Such a sequence of transpositions can be reordered in the form $(e_1\ a), \dots, (e_l\ a), (e_{l+1}\ b), \dots, (e_{n-1}\ b)$ with $e_i \in C_a^- \cup C_b^+$ if $i \leq l$ and $e_i \in C_a^+ \cup C_b^-$ if $i \geq l + 1$. Therefore the two sets $C_a^- \cup C_b^+ \cup \{a\}$ e $C_a^+ \cup C_b^- \cup \{b\}$ are disjoint, non-empty and closed with respect to the action of the group generated by these transpositions. Of course, this fact contradicts the connection of $p^\gamma \cong p^\beta$. So, γ cannot meet any β_i in some point other than $*$. $\square$

Lemma 4.2. *Let β be a curve such that $\beta = (\alpha_m)b$, with $b \in \mathcal{L}_p$ and $1 \leq m \leq n$, and $p^\beta \cong p^{\alpha_m}$ is connected. Then β is $\mathcal{I}_p$-equivalent to a curve δ such that $\delta \cap \alpha_i = \{*\}$ for some $i = 1, \dots, m$ and δ starts from $*$ on the right (resp. left) of α_i if $i < m$ (resp. $i \geq m$).*

Proof. By Lemma 4.1, β is $\mathcal{I}_p$-equivalent to a curve γ which meets at least one of the α_i's only in $*$. In other words, the set $S \subset \{1, \dots, n\}$ of the i's such that $\gamma \cap \alpha_i = \{*\}$ is nonempty. We can also assume that γ has minimum index (with respect to the fundamental system $\alpha_1, \dots, \alpha_n$) among all the curves having such property in the $\mathcal{I}_p$-equivalence class of β.

If there exists $i \in S$ such that either $i < m$ and γ starts from $*$ on the left of α_i or $i \geq m$ and γ starts from $*$ on the right of α_i, then we can put $\delta = \gamma$.

If such an i does not exist, but there exists $i \in S$ such that the interval $x \simeq \gamma \cup \alpha_i$ is of type 1 or 2, then we can put $\delta = (\gamma)x^{\pm 2}$, with $+$ or $-$ depending on the fact that γ starts from $*$ on the left or on the right of α_i.

In the remaining cases, all the curves α_i with $i \in S$ have the same monodromy and start from $*$ on the same side with respect to γ. Assuming this property and also that γ minimizes the number of intersection points with $\alpha_1 \cup \ldots \cup \alpha_n$ in its isotopy class, we construct the curves $\gamma_1, \dots, \gamma_n$ as in the proof of Lemma 4.1 with the α_i's in place of the β_i. In particular we get $\gamma_i = \alpha_i$ if $i \in S$. At this point, we can carry on the proof analogously to the proof of Lemma 4.1, with the only difference that, each time a curve γ_i with $i \in S$ is involved in the reasoning, we get a good definition of δ instead of a contradiction with respect to the minimality of γ. $\square$

Proof of Theorem B. We proceed by induction on the number n of branch points of p. For $n = 1$ the result is trivial. So, let us suppose $n > 1$.

On the other hand, the case $p \cong p_n$ has been examined in Lemma 3.10. Hence we can also assume $p \ncong p_n$, in such a way that there exists $m \leq d-1$ minimum index such that $\varphi(\alpha_m) = \varphi(\alpha_{m+1})$. Then p^{α_m} is connected and $\varphi(\alpha_m) = \varphi(\alpha_{m+1}) = (m\ m{+}1)$.

We start by observing that, if $b \in \mathcal{L}_p$ and there exists a curve α for p such that p^α is connected and $(\alpha)b$ is $\mathcal{I}_p$-equivalent to α, then $b \in \mathcal{I}_p$. Indeed, if $c \in \mathcal{I}_p$ is such that $(\alpha)b = (\alpha)c$, then $(\alpha)bc^{-1} = \alpha$ and therefore bc^{-1} can be thought as a braid in $\mathcal{L}_{p^\alpha}$. By the induction hypothesis, we have $bc^{-1} \in \mathcal{I}_{p^\alpha} \subset \mathcal{I}_p$ and therefore $b \in \mathcal{I}_p$. It is easy to see that an analogous argument also holds if p^α is not connected but has at most one non-trivial component.

Now, let $b \in \mathcal{L}_p$ be an arbitrary liftable braid. By Lemma 4.2, the curve $\beta = (\alpha_m)b$ is $\mathcal{I}_p$-equivalent to a curve γ such that $\gamma \cap \alpha_i = \{*\}$ for some $i = 1, \dots, n$. Moreover, γ starts from $*$ on the right of α_i if $i < m$ and on the left of α_i if $i \geq m$. At this point, we conclude the proof by checking separately the three possible cases.

(1) $i < m$. In this case $\varphi(\alpha_i) = (i\ i{+}1)$ and both the restrictions p^{α_i} and p^{α_i,α_m} have two components, one of which is trivial (the one corresponding to the sheet $i+1$ with respect to the base point $*'$). On the other hand, p^γ is connected and therefore the components of $p^{\alpha_i,\gamma}$ can not be more than two and they coincide with the ones of p^{α_i}. By Lemma 2.2, there exists $c \in \mathcal{L}_p$ such that $(\alpha_i)c = \alpha_i$ and $(\alpha_m)c = \gamma$. By applying the induction hypothesis to c thought as a braid in $\mathcal{L}_{p^{\alpha_i}}$, we have that $c \in \mathcal{I}_{p^{\alpha_i}} \subset \mathcal{I}_p$ and therefore $\beta = (\alpha_m)b$ is $\mathcal{I}_p$-equivalent to α_m. Finally, the starting observation enable us to conclude that $b \in \mathcal{I}_p$.

(2) $i = m, m+1$ or $i > m+1$ with $m = d-1$. In this case the interval $x \simeq \gamma \cup \alpha_i$ is of type 1 and γ is $\mathcal{I}_p$-equivalent to α_i and therefore to α_m. Then $b \in \mathcal{I}_p$, since $\beta = (\alpha_m)b$ is $\mathcal{I}_p$-equivalent to α_m.

(3) $i > m+1$ with $m < d-1$. In this case we have $\varphi(\alpha_i) = (l\ l{+}1)$ with $l > m$, moreover the restrictions p^{α_i} and p^{α_m,α_i} are both connected or they have two components one of which is trivial (the one corresponding to the sheet $i+1$ with respect to the base point $*'$). We consider a fundamental system $\delta_1, \dots, \delta_{n-2}, \gamma, \alpha_i$ for p and set $\varphi(\delta_j) = \sigma_j$ for each $j = 1, \dots, n-2$. Then $\sigma_1 \dots \sigma_{n-2} = \varphi(\omega)\,(l\ \ l{+}1)\,(m\ \ m{+}1) = (m\ \ m{-}1\ \ \dots\ \ 1)\,\sigma(l\ \ l{+}1)\,(m\ \ m{+}1)$ with σ product of cycles all disjoint from $(m\ \ m{-}1\ \ \dots\ \ 1)$. It follows that $(\sigma_1 \dots \sigma_{n-2})^m(m) = m+1$. Hence the orbits of the action of $\langle \sigma_1, \dots, \sigma_{n-2} \rangle \subset \Sigma_d$ coincide with the ones of the action of $\langle \sigma_1, \dots, \sigma_{n-2}, (m\ m{+}1) \rangle$, so that also the components of p^{γ,α_i} correspond to the ones of p^{α_i}. By Lemma 2.2, there exists $c \in \mathcal{L}_p$ such that $(\alpha_m)c = \gamma$ and $(\alpha_i)c = \alpha_i$. Then, we can conclude that $b \in \mathcal{I}_p$ by the same argument of case (1).

At this point, in order to prove that $\mathcal{L}_p$ is finitely generated and therefore can be generated by a finite set of liftable powers of intervals, it suffices to observe that $\mathcal{L}_p$ is a subgroup of finite index of $\mathcal{B}_n$ (see [8]). In fact, given $b, c \in \mathcal{B}_n$, we have that $bc^{-1} \in \mathcal{L}_p$ if and only if $\varphi((\alpha_i)bc^{-1}) = \varphi(\alpha_i)$ for every $i = 1, \dots, n$, by Lemma 2.1. Then b and c belong to the same coset of $\mathcal{L}_p$ in $\mathcal{B}_n$ if and only if $\varphi((\alpha_i)b) = \varphi((\alpha_i)c)$ for every $i = 1, \ldots, n$. This means that there is a bijective correspondence between cosets of $\mathcal{L}_p$ in $\mathcal{B}_n$ and admissible sequences of transpositions of Σ_d of length n. Therefore $|\,\mathcal{B}_n : \mathcal{L}_p\,| \leq (d(d-1)/2)^n$. $\square$

References

[1] I. Berstein and A. L. Edmonds, *On the construction of branched coverings of low-dimensional manifolds*, Trans. Amer. Math. Soc. **247** (1979), 87–124.

[2] I. Berstein and A. L. Edmonds, *On the classification of generic branched coverings of surfaces*, Illinois J. Math. **28** (1984), 64–82.

[3] J. S. Birman, *Braids, Link, and Mapping Class Groups*, Ann. of Math. Studies **82**, Princeton University Press 1974.

[4] J. S. Birman and B. Wajnryb, *3-fold branched coverings and the mapping class group of a surface*, in "Geometry and Topology", Lecture Notes in Math. **1167**, Springer-Verlag 1985, 24–46.

[5] J. S. Birman and B. Wajnryb, *Presentations of the mapping class group. Errata: "3-fold branched coverings and the mapping class group of a surface" and "A simple presentation of the mapping class group of an orientable surface"*, Israel J. Math. **88** (1994), 425–427.

[6] D. Gabai and W. H. Kazez, *The classification of maps of surfaces*, Invent. Math. **90** (1987), 219–242.

[7] D. Gabai and W. H. Kazez, *The Classification of Maps of Nonorientable Surfaces*, Math. Ann. **281** (1988), 687–702.

[8] W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*, Pure and Applied Mathematics, Vol. **XIII**, Interscience Publishers 1966.

[9] R. Piergallini, *Covering Moves*, Trans Amer. Math. Soc. **325** (1991), 903–920.

[10] R. Piergallini, *Four-manifolds as 4-fold branched covers of S^4*, Topology **34** (1995), 497-508.