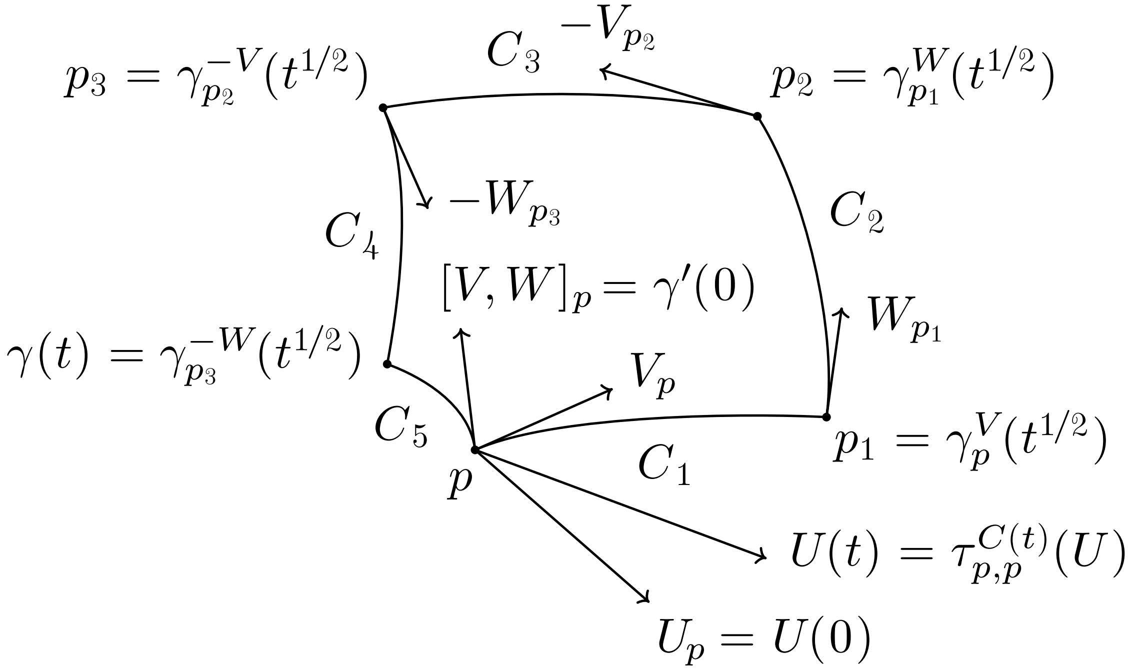
Tensore di curvatura

(M,g) varietà riemanniana
$\rightsquigarrow R : \mathcal{V}M \times \mathcal{V}M \to \operatorname{End}\mathcal{V}M$ applicazione tale che
$$R(V,W)(U) \overset{\text{def}}{=\!=} \nabla_V \nabla_W U - \nabla_W \nabla_V U - \nabla_{[V,W]} U \quad \forall\, V,W,U \in \mathcal{V}M$$

Note: 1) R applicazione $\mathbb{R}$-bilineare antisimmetrica ben definita ($R(V,W)$ $\mathbb{R}$-lineare e $R(W,V) = -R(V,W)$ $\forall\, V,W \in \mathcal{V}M$)
2) $R(V,W)(U)$ invar. per similitudini locali (poiché lo è ∇)
3) $R(V,W)(U)$ ha la seguente interpretazione geometrica
$$R(V,W)(U)_p = U'(0) = \lim_{t\to 0}(\tau_{p,p}^{C(t)=C_1\cup\ldots\cup C_5}(U_p) - U_p)/t$$



Prop. (M,g) varietà riemanniana $\Rightarrow$
1) $R : \mathcal{V}M \times \mathcal{V}M \to \operatorname{End}\mathcal{V}M$ appl. $C^\infty(M)$-bilineare
2) $R(V,W) : \mathcal{V}M \to \mathcal{V}M$ appl. $C^\infty(M)$-lineare $\forall\, V,W \in \mathcal{V}M$
3) $R(V,W)(U) + R(W,U)(V) + R(U,V)(W) = 0 \ \forall\, V,W,U \in \mathcal{V}M$
identità di Bianchi

Dim. 1)
$$\begin{aligned} R(fV,W)(U) &= \nabla_{fV}\nabla_W U - \nabla_W \nabla_{fV} U - \nabla_{[fV,W]} U \\ &= f\nabla_V \nabla_W U - \nabla_W f \nabla_V U - \nabla_{f[V,W]-(Wf)V} U \\ &= f\nabla_V\nabla_W U - (Wf)\nabla_V U + f\nabla_W\nabla_V U \\ &\quad - f\nabla_{[V,W]}U + (Wf)\nabla_V U = fR(V,W)(U) \end{aligned}$$
$R(V,fW)(U) = fR(V,W)(U)$ (segue per l'antisimmetria)

2) $R(V,W)(fU) = \nabla_V \nabla_W (fU) - \nabla_W \nabla_V (fU) - \nabla_{[V,W]}(fU)$
$= \nabla_V((Wf)U + f\nabla_W U) - \nabla_W((Vf)U + f\nabla_V U)$
$- ([V,W]f)U - f\nabla_{[V,W]}U$
$= (VWf)U + (Wf)\nabla_V U + (Vf)\nabla_W U + f\nabla_V \nabla_W U$
$- (WVf)U - (Vf)\nabla_W U - (Wf)\nabla_V U - f\nabla_W \nabla_V U$
$- ([V,W]f)U - f\nabla_{[V,W]}U = fR(V,W)(U)$

3) $R(V,W)(U) + R(W,U)(V) + R(U,V)(W)$
$= \nabla_V \nabla_W U - \nabla_W \nabla_V U - \nabla_{[V,W]}U$
$+ \nabla_W \nabla_U V - \nabla_U \nabla_W V - \nabla_{[W,U]}V$
$+ \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]}W$
$= \nabla_V [W,U] + \nabla_W [U,V] + \nabla_U [V,W]$
$- \nabla_{[V,W]}U - \nabla_{[W,U]}V - \nabla_{[U,V]}W$
$= [V,[W,U]] + [W,[U,V]] + [U,[V,W]] = 0$

Note: 1) R è l'applicazione nulla se $\dim M < 2$

2) $(F_1, \ldots, F_m)$ riferimento locale
$V = \sum_i v^i F_i$, $W = \sum_j w^j F_j$, $U = \sum_k u^k F_k$
$\Rightarrow R(V,W)(U) = \sum_{i,j,k} v^i w^j u^k R(F_i, F_j)(F_k)$

3) $R_{ijk} \overset{\text{def}}{=\!=} R(F_i, F_j)(F_k)$ dipende solo dal riferimento
$\rightsquigarrow R_{ijk} = \sum_h R^h_{ijk} F_h$, $R_{ijkh} = g(R_{ijk}, F_h)$
t.c. $R_{ijkh} = \sum_\ell g_{h\ell} R^\ell_{ijk}$, $R^h_{ijk} = \sum_\ell g^{h\ell} R_{ijk\ell}$

4) in coordinate locali $(F_1 = \partial/\partial x^1, \ldots, F_m = \partial/\partial x^m)$

$$R_{ijk} = \nabla_{\frac{\partial}{\partial x^i}} \Gamma_{jk} - \nabla_{\frac{\partial}{\partial x^j}} \Gamma_{ik} = \nabla_{\frac{\partial}{\partial x^i}} \sum_h \Gamma^h_{jk} \frac{\partial}{\partial x^h} - \nabla_{\frac{\partial}{\partial x^j}} \sum_h \Gamma^h_{ik} \frac{\partial}{\partial x^h}$$
$$= \sum_h \Big(\frac{\partial \Gamma^h_{jk}}{\partial x^i} \frac{\partial}{\partial x^h} - \frac{\partial \Gamma^h_{ik}}{\partial x^j} \frac{\partial}{\partial x^h} + \Gamma^h_{jk} \Gamma_{ih} - \Gamma^h_{ik} \Gamma_{jh} \Big)$$
$$= \sum_h \Big(\frac{\partial \Gamma^h_{jk}}{\partial x^i} - \frac{\partial \Gamma^h_{ik}}{\partial x^j} + \sum_\ell (\Gamma^\ell_{jk} \Gamma^h_{i\ell} - \Gamma^\ell_{ik} \Gamma^h_{j\ell}) \Big) \frac{\partial}{\partial x^h}$$
$$\rightsquigarrow R^h_{ijk} = \frac{\partial \Gamma^h_{jk}}{\partial x^i} - \frac{\partial \Gamma^h_{ik}}{\partial x^j} + \sum_\ell (\Gamma^\ell_{jk} \Gamma^h_{i\ell} - \Gamma^\ell_{ik} \Gamma^h_{j\ell})$$
$$R_{ijkh} = \sum_\ell \Big(g_{h\ell} \Big(\frac{\partial \Gamma^\ell_{jk}}{\partial x^i} - \frac{\partial \Gamma^\ell_{ik}}{\partial x^j} \Big) + \Gamma^\ell_{jk} \Gamma_{i\ell h} - \Gamma^\ell_{ik} \Gamma_{j\ell h} \Big)$$

5) $R(V,W)(U)_p$ dipende solo da V_p, W_p, U_p $\forall\, p \in M$

(M, g) varietà riemanniana, $p \in M$

$\rightsquigarrow R_p : T_pM \times T_pM \to \operatorname{End} T_pM$ appl. bilineare antisimm. t.c.

$R_p(v, w)(u) \stackrel{\text{def}}{=\!=} R(V, W)(U)_p \ \forall\, v, w, u \in T_pM \ \forall\, V, W, U \in \mathcal{V}M$ con $V_p = v$, $W_p = w$, $U_p = u$

↳ operatore di curvatura associato alla coppia (v, w)

$\rightsquigarrow R_p : (T_pM)^4 \to R$ forma multilineare tale che

$R_p(v, w, u, t) \stackrel{\text{def}}{=\!=} g_p(R_p(v, w)(u), t)$ per ogni $v, w, u, t \in T_pM$

↳ tensore di curvatura di Riemann

Prop. (M, g) varietà riemanniana, $p \in M$

$$\left.\begin{array}{l} 1)\ R_p(w, v, u, t) = -R_p(v, w, u, t) \\ 2)\ R_p(v, w, t, u) = -R_p(v, w, u, t) \\ 3)\ R_p(u, t, v, w) = R_p(v, w, u, t) \end{array}\right\} \forall\, v, w, u, t \in T_pM$$

Dim. 1) segue dalla antisimmetria di R

2) basta provare $R_p(v, w, u, u) = 0 \ \forall\, v, w, u \in T_pM$ (multilin.)

quindi $g(R(V, W)(U), U) = 0 \ \forall\, V, W, U \in \mathcal{V}M$ (def. di R_p)

$$\begin{aligned} g(R(V, W)(U), U) &= \\ &= g(\nabla_V \nabla_W U, U) - g(\nabla_W \nabla_V U, U) - g(\nabla_{[V,W]} U, U) \\ &= V g(\nabla_W U, U) - g(\nabla_W U, \nabla_V U) \\ &\quad - W g(\nabla_V U, U) + g(\nabla_V U, \nabla_W U) - \frac{1}{2}[V, W] g(U, U) \\ &= \frac{1}{2} V W g(U, U) - \frac{1}{2} W V g(U, U) - \frac{1}{2}[V, W] g(U, U) = 0 \end{aligned}$$

3) definizione di R_p + identità di Bianchi

$$\begin{aligned} \Rightarrow\ & R_p(v, w, u, t) + R_p(w, u, v, t) + R_p(u, v, w, t) = 0 \\ & R_p(w, u, t, v) + R_p(u, t, w, v) + R_p(t, w, u, v) = 0 \\ & R_p(u, t, v, w) + R_p(t, v, u, w) + R_p(v, u, t, w) = 0 \\ & R_p(t, v, w, u) + R_p(v, w, t, u) + R_p(w, t, v, u) = 0 \\ \Rightarrow\ & 2R_p(v, w, u, t) - 2R_p(u, t, v, w) = 0 \\ \Rightarrow\ & R_p(u, t, v, w) = R_p(v, w, u, t) \ \forall\, v, w, u, t \in T_pM \end{aligned}$$

Prop. (M, g) varietà riemanniana, $p \in M$

R_p è univoc. determinato da $R_p(v, w, v, w) \ \forall\, v, w \in T_pM$

Dim. $R_p(v, w, v, w) \ \forall\, v, w \in T_pM \rightsquigarrow R_p(v, w, u, w) \ \forall\, v, w, u \in T_pM$

$$(R_p(v+u,w,v+u,w) = R_p(v,w,v,w) + R_p(v,w,u,w) + R_p(u,w,v,w) + R_p(u,w,u,w)$$

$$\rightsquigarrow R_p(v,w,u,w) = (R_p(v+u,w,v+u,w) - R_p(v,w,v,w) - R_p(u,w,u,w))/2)$$

$$R_p(v,w,u,w)\ \forall\, v,w,u \in T_pM \rightsquigarrow R_p(v,w,u,t)\ \forall\, v,w,u,t \in T_pM$$

$$(R_p(v,w+t,u,w+t) = R_p(v,w,u,w) + R_p(v,w,u,t) + R_p(v,t,u,w) + R_p(v,t,u,t)$$

$$\rightsquigarrow R_p(v,w,u,t) - R_p(w,u,v,t) = a$$

con $a = R_p(v,w+t,u,w+t) - R_p(v,w,u,w) - R_p(v,t,u,t))$

$$\rightsquigarrow R_p(w,u,v,t) - R_p(u,v,w,t) = b$$

con $b = R_p(w,u+t,v,u+t) - R_p(w,u,v,u) - R_p(w,t,v,t)$

$$\rightsquigarrow \begin{cases} R_p(v,w,u,t) - R_p(w,u,v,t) = a \\ R_p(w,u,v,t) - R_p(u,v,w,t) = b \\ R_p(v,w,u,t) + R_p(w,u,v,t) + R_p(u,v,w,t) = 0 \end{cases}$$

$$\rightsquigarrow R_p(v,w,u,t) = (2a+b)/3)$$

<u>Note</u>: 1) $\{R_p\}_{p\in M} \leftrightsquigarrow R : (\mathcal{V}M)^4 \to C^\infty(M)$ forma multilineare t.c. $R(V,W,U,T)(p) \stackrel{\text{def}}{=\!=} R_p(V_p,W_p,U_p,T_p)\ \forall\, p \in M$, cioè

$$R(V,W,U,T) = g(R(V,W)(U),T)\ \forall\, V,W,U,T \in \mathcal{V}M$$

2) $(F_1,\ldots,F_m)$ riferimento locale intorno a $p \in M$

$V = \sum_i v^i F_i$, $W = \sum_j w^j F_j$, $U = \sum_k u^k F_k$, $T = \sum_h t^h F_h$

$\Rightarrow R(V,W,U,T) = \sum_{ijkh} R_{ijkh}\, v^i w^j u^k t^h$

con $R_{ijkh} = R(F_i,F_j,F_k,F_h)\ \ \forall\, i,j,k,h = 1,\ldots,m$

3) $$\left.\begin{array}{l} R_{jihk} = R_{hkji} = R_{khij} = R_{ijkh} \\ R_{jikh} = R_{ijhk} = R_{khji} = R_{hkij} = -R_{ijkh} \end{array}\right\}\forall\, i,j,k,h = 1,\ldots,m$$

$\Rightarrow R_{iijk} = R_{ijkk} = 0\ \forall\, i,j,k = 1,\ldots,m$

(M,g) varietà riemanniana, $p \in M$

$\rightsquigarrow S_{v,w}(u,t) \stackrel{\text{def}}{=\!=} (R_p(v,u,w,t) + R_p(v,t,w,u))/2$

forma bilineare simmetrica per ogni $v,w \in T_pM$

$\rightsquigarrow S_p : T_pM \times T_pM \to R$ forma bilineare simmetrica tale che $S_p(v,w) \stackrel{\text{def}}{=\!=} \mathrm{tr}(S_{v,w})$ (teorema spettrale)

$\hookrightarrow$ <u>tensore di curvatura di Ricci</u>

<u>Note</u>: 1) $\{S_p\}_{p\in M} \leftrightsquigarrow S : \mathcal{V}M \times \mathcal{V}M \to C^\infty(M)$ forma bilineare
t.c. $S(V,W)(p) \overset{\text{def}}{=\!=} S_p(V_p,W_p)\ \forall\, V,W \in \mathcal{V}M\ \forall\, p\in M$

2) $(F_1,\dots,F_m)$ riferimento locale intorno a $p\in M$
$V = \sum_i v^i F_i\,,\ W = \sum_j w^j F_j \Rightarrow S(V,W) = \sum_{ij} S_{ij} v^i w^j$
con $S_{ij} = S(F_i,F_j) = \sum_k (R^k_{ikj} + R^k_{jki})/2\ \ \forall\, i,j = 1,\dots,m$
$(g(\varphi_{ij}(F_k),F_h) = S_{F_i,F_j}(F_k,F_h) \Rightarrow S_{ij} = \sum_k \varphi_{ij}(F_k)^k)$

<u>Prop</u>. $f : (M,g^M) \to (N,g^N)$ isometria/similitudine (locale)
con fattore di similitudine s ($= 1$ se f isometria (locale))
$\Rightarrow R^N_{f(p)}(T_pf(v),T_pf(w),T_pf(u),T_pf(t)) = s^2 R^M_p(v,w,u,t)$
$S^N_{f(p)}(T_pf(v),T_pf(w)) = S^M_p(v,w)\ \ \forall\, v,w,u,t \in T_pM$

<u>Dim</u>. ∇ invar. per simil. (locali) $\Rightarrow R(\cdot\,,\cdot)$ invar. per simil. (locali)
$\Rightarrow g^N_{f(p)}(R^N_{f(p)}(\cdot\,,\cdot)(\cdot),\cdot) = s^2 g^M_p(R^M_p(\cdot\,,\cdot)(\cdot),\cdot)$
$S^N_{f(p)}(\cdot\,,\cdot) = \mathrm{tr}(S^N_{(\cdot,\cdot)}(\cdot\,,\cdot)) = \mathrm{tr}(S^M_{(\cdot,\cdot)}(\cdot\,,\cdot)) = S^M_p(\cdot\,,\cdot)$

<u>Curvature</u>

(M,g) varietà riemanniana, $\dim M \geq 2$

$\sigma = \langle v_1,v_2\rangle \subset T_pM$ <u>sezione piana</u> (v_1 e v_2 lin. indip.)

$$\rightsquigarrow K(\sigma) \overset{\text{def}}{=\!=} -\frac{R_p(v_1,v_2,v_1,v_2)}{\mathrm{Area}(v_1,v_2)^2} = -\frac{R_p(v_1,v_2,v_1,v_2)}{g_p(v_1,v_1)\,g_p(v_2,v_2) - g_p(v_1,v_2)^2}$$

↖ <u>curvatura sezionale</u> della sezione σ

<u>Note</u>: 1) $K(\sigma)$ è ben definita, non dipende dalla base (v_1,v_2)
$(\sigma = \langle w_1,w_2\rangle \Rightarrow (w_1,w_2) = M(v_1,v_2)$
$\Rightarrow R_p(w_1,w_2,w_1,w_2) = \det M^2\, R_p(v_1,v_2,v_1,v_2)$
$\mathrm{Area}(w_1,w_2) = \det M\, \mathrm{Area}(v_1,v_2))$

2) (v_1,v_2) base ortonormale di $\sigma \Rightarrow K(\sigma) = -R(v_1,v_2,v_1,v_2)$

3) $K(\sigma)\ \forall\,\sigma \subset T_pM \rightsquigarrow R_p(v,w,u,t)\ \forall\, v,w,u,t \in T_pM$

4) $m = 2 \Rightarrow \sigma = T_pM \rightsquigarrow K(\sigma) = K(p)$ curvatura di Gauss

$\lambda = \langle v\rangle \subset T_pM$ <u>direzione tangente</u> ($v \neq 0$)

$$\rightsquigarrow K(\lambda) \overset{\text{def}}{=\!=} -\frac{1}{m-1}\,\frac{S_p(v,v)}{\|v\|^2}$$

↖ <u>curvatura di Ricci</u> nella direzione λ

$$p \in M \rightsquigarrow K(p) = -\frac{1}{m(m-1)}\,\mathrm{tr}(S_p) \text{ (teorema spettrale)}$$

↖ <u>curvatura scalare</u> nel punto p

<u>Note</u>: 1) $K(\lambda)$ è ben definita, non dipende dal vettore v

$(\lambda = \langle w \rangle \Rightarrow w = kv \Rightarrow S_p(w,w) = k^2 S_p(v,v), \|w\|^2 = k^2\|v\|)$

2) $\lambda = \langle v \rangle$ con $v \in T_pM$ versore

$\{v_1, \ldots, v_m\}$ base ortonorm. di T_pM tale che $v_m = v$

$\Rightarrow K(\lambda) = \frac{1}{m-1} \sum_{i=1}^{m-1} K(\sigma_{im})$ con $\sigma_{im} = \langle v_i, v_m \rangle = \langle v_i, v \rangle$

3) $\{v_1, \ldots, v_m\}$ base ortonorm. di T_pM

$K(p) = \frac{1}{m} \sum_{i=1}^{m} K(\lambda_i)$ con $\lambda_i = \langle v_i \rangle$

$= \frac{2}{m(m-1)} \sum_{i<j} K(\sigma_{ij})$ con $\sigma_{ij} = \langle v_i, v_j \rangle$

4) $m = 2 \Rightarrow K(\lambda) = K(p)$ curvatura di Gauss $\forall\, \lambda \subset T_pM$

$m = 3 \Rightarrow K(\lambda)\ \forall\, \lambda \subset T_pM \rightsquigarrow K(\sigma)\ \forall\, \sigma \subset T_pM$

$(\{v_1, v_2, v_3\}$ base ortonorm. di T_pM

$$\rightsquigarrow \begin{cases} K(\sigma_{12}) + K(\sigma_{13}) = 2K(\lambda_1) \\ K(\sigma_{12}) + K(\sigma_{23}) = 2K(\lambda_2) \\ K(\sigma_{13}) + K(\sigma_{23}) = 2K(\lambda_3) \end{cases}$$

$\rightsquigarrow K(\sigma_{12}) = K(\lambda_1) + K(\lambda_2) - K(\lambda_3))$

<u>Prop</u>. (M, g) varietà riemanniana, $\sigma \subset T_pM$ sezione piana

$\Rightarrow K(\sigma) = K^S(p)$ con $S = S_\sigma = e_p(\sigma \cap B(0, \varepsilon_p)) \subset M$

<u>Dim</u>. $(x^1, \ldots, x^m)$ coord. norm. su (A_p, φ_p) t.c. $\sigma = \langle \partial/\partial x^1, \partial/\partial x^2 \rangle$

$\rightsquigarrow (F_1, \ldots, F_m)$ riferimento ortonormale tale che $\langle F_1(q), F_2(q) \rangle = \langle \partial/\partial x^1, \partial/\partial x^2 \rangle = T_qS\ \ \forall\, q \in S$ } (G.-S.)

per $x = 0$ (cioè in p) e $i, j = 1, 2$ si ha:

$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$ $(\gamma_v(t) = tv$ geodetica $\forall\, v = \sum_k v^k \partial/\partial x^k \in T_pM$

$\Rightarrow \nabla_v \gamma_v' = \sum_{i,j} v^i v^j \Gamma_{ij} = 0\ \forall\, (v^1, \ldots, v^m) \in R^m)$

$$\Rightarrow \nabla_{F_i} F_j \in T_pS \ \left(\nabla_{F_i} F_j = \sum_{k,h} a_i^k\, \nabla_{\frac{\partial}{\partial x^k}} \left(a_j^h \frac{\partial}{\partial x^h} \right) = \sum_{k,h} a_i^k \frac{\partial a_j^h}{\partial x^k} \frac{\partial}{\partial x^h} \right)$$

$$\Rightarrow g(\nabla^S_{F_i}\nabla^S_{F_j}F_1, F_2) = g(\nabla_{F_i}\nabla^S_{F_j}F_1, F_2)$$
$$= F_i\, g(\nabla^S_{F_j}F_1, F_2) - g(\nabla^S_{F_j}F_1, \nabla_{F_i}F_2)$$
$$= F_i\, g(\nabla_{F_j}F_1, F_2) - g(\nabla_{F_j}F_1, \nabla_{F_i}F_2) = g(\nabla_{F_i}\nabla_{F_j}F_1, F_2)$$
$$\Rightarrow R^S_{1212} = g(\nabla^S_{F_1}\nabla^S_{F_2}F_1 - \nabla^S_{F_2}\nabla^S_{F_1}F_1 - \nabla^S_{[F_1,F_2]}F_1, F_2)$$
$$= g(\nabla_{F_1}\nabla_{F_2}F_1 - \nabla_{F_2}\nabla_{F_1}F_1 - \nabla_{[F_1,F_2]}F_1, F_2) = R_{1212}$$

Corol. $(N, g^N) \subset (M, g^M)$ sottovarietà riemann.

$C^N_v \subset C^M_v \ \forall\, v \in T_pN$ (sottovarietà geodetica in p)

$\Rightarrow K^N(\sigma) = K^M(\sigma)$ per ogni sezione piana $\sigma \subset T_pN$

Dim. $C^N_v \subset C^M_v \ \forall\, v \in T_pN \Rightarrow e^N_{p|B(0,\varepsilon)}$ restrizione di $e^M_{p|B(0,\varepsilon)}$

$\Rightarrow S^N_\sigma = e^N_p(\sigma \cap B(0,\varepsilon)) = e^M_p(\sigma \cap B(0,\varepsilon)) = S^M_\sigma$

$\Rightarrow K^N(\sigma) = K^{S^N_\sigma}(p) = K^{S^M_\sigma}(p) = K^M(\sigma)$

Nota: *in particolare il corollario vale per* N sottovarietà totalmente geodetica (geodetica *in* p per ogni $p \in N$) (per esempio $N = \operatorname{Fix} H$ con $H \subset \operatorname{Isom} M$)

Esempi: 1) $R^m \rightsquigarrow g_{ij} = \delta_{ij} \Rightarrow R_{ijk\ell} = 0$ per ogni i, j, k, ℓ

$\Rightarrow K(\sigma) = K(\lambda) = K(p) = 0 \ \forall\, \sigma, \lambda \subset T_pR^m,\ p \in R^m$

2) $g_{ij} = s(x)^2\delta_{ij}$ (metrica conformemente piatta) $\Rightarrow$

$$R_{ijij} = s\Big(\frac{\partial^2 s}{(\partial x^i)^2} + \frac{\partial^2 s}{(\partial x^j)^2}\Big) - \Big(\frac{\partial s}{\partial x^i}\Big)^2 - \Big(\frac{\partial s}{\partial x^j}\Big)^2 + \sum_{k\neq i,j}\Big(\frac{\partial s}{\partial x^k}\Big)^2$$

3) $s(x)$ dipendente solo da x^m

$$\Rightarrow \begin{cases} R_{ijij} = \Big(\dfrac{\partial s}{\partial x^m}\Big)^2 & \text{per ogni } i < j < m \\[2ex] R_{imim} = s\dfrac{\partial^2 s}{(\partial x^m)^2} - \Big(\dfrac{\partial s}{\partial x^m}\Big)^2 & \text{per ogni } i < m \end{cases}$$

4) $H^m \rightsquigarrow s(x) = 1/x^m$

$\Rightarrow R_{ijij} = 1/(x^m)^4$ per ogni $i < j$

$\Rightarrow K(\sigma) = K(\lambda) = K(p) = -1 \ \forall\, \sigma, \lambda \subset T_pH^m,\ p \in H^m$

5) $s(x)$ dipendente solo da ρ^2

$$\Rightarrow \frac{\partial s}{\partial x^i} = 2x^i\frac{\partial s}{\partial \rho^2}\ ,\ \ \frac{\partial^2 s}{(\partial x^i)^2} = 2\frac{\partial s}{\partial \rho^2} + 4(x^i)^2\frac{\partial^2 s}{(\partial \rho^2)^2}$$

$$\Rightarrow R_{ijij} = 4s\Big(\frac{\partial s}{\partial \rho^2} + \big((x^i)^2+(x^j)^2\big)\frac{\partial^2 s}{(\partial\rho^2)^2}\Big) + 4\big(\rho^2 - 2(x^i)^2 - 2(x^j)^2\big)\Big(\frac{\partial s}{\partial\rho^2}\Big)^2$$

6) $S^m, D^m \rightsquigarrow s(\rho^2) = 2/(1\pm\rho^2)$

$\Rightarrow \partial s/\partial\rho^2 = \mp 2/(1\pm\rho^2)^2\ ,\ \partial^2 s/(\partial\rho^2)^2 = 4/(1\pm\rho^2)^3$

$\Rightarrow R_{ijij} = \mp 16/(1\pm\rho^2)^4$

$\Rightarrow K(\sigma) = K(\lambda) = K(p) = +1\ \ \forall\,\sigma,\lambda\subset T_pS^m,\ p\in S^m$

$K(\sigma) = K(\lambda) = K(p) = -1\ \ \forall\,\sigma,\lambda\subset T_pD^m,\ p\in D^m$

Prop. $f:(M,g^M)\to(N,g^N)$ isometria/similitudine (locale) con fattore di similitudine s ($=1$ se f isometria (locale))

$\Rightarrow K^N(T_pf(\sigma)) = K^M(\sigma)/s^2$ per ogni sezione piana $\sigma\subset T_pM$

$K^N(T_pf(\lambda)) = K^M(\lambda)/s^2$ per ogni direzione $\lambda\subset T_pM$

$K^N(f(p)) = K^M(p)/s^2$ per ogni punto $p\in M$

Dim. $R^N_{f(p)}(\cdot,\cdot,\cdot,\cdot) = s^2\,R^M_p(\cdot,\cdot,\cdot,\cdot)\,,\ \mathrm{Area}^N_{f(p)}(\cdot,\cdot) = s^2\mathrm{Area}^M_p(\cdot,\cdot)$

$\Rightarrow$ la relazione vale per la curvatura sezionale

$\Rightarrow$ vale per le curvature di Ricci e scalare

Esempi: 1) $S^m_r \overset{\text{def}}{=\!=} rS^m\subset R^{m+1}$ sfera di raggio $r>0$ con la metrica indotta da R^{m+1}

$\rightsquigarrow ds^2 = \dfrac{4r^4}{(r^2+\|x\|^2)^2}\sum_i(dx^i)^2$ in coord. stereog. su S^m_r

$\cong \dfrac{4r^2}{(1+\|x\|^2)^2}\sum_i(dx^i)^2$ in coord. stereog. su S^m

$\Rightarrow K(\sigma) = K(\lambda) = K(p) = 1/r^2\ \ \forall\,\sigma,\lambda\subset T_pS^m_r,\ p\in S^m_r$

2) $H^m_r \overset{\text{def}}{=\!=} \big(\mathrm{Int}\,R^m_+, ds^2 = r^2/(x^m)^2\sum_i(dx^i)^2\big)$ con $r>0$

$\Rightarrow K(\sigma)=K(\lambda)=K(p) = -1/r^2\ \forall\,\sigma,\lambda\subset T_pH^m_r,\ p\in H^m_r$

3) $D^m_r \overset{\text{def}}{=\!=} \Big(\mathrm{Int}\,rB^m, ds^2 = \dfrac{4r^4}{(r^2-\|x\|^2)^2}\sum_i(dx^i)^2\Big)$

$D^m_r \cong \Big(\mathrm{Int}\,B^m, ds^2 = \dfrac{4r^2}{(1-\|x\|^2)^2}\sum_i(dx^i)^2\Big)$

$\Rightarrow K(\sigma)=K(\lambda)=K(p) = -1/r^2\ \forall\,\sigma,\lambda\subset T_pD^m_r,\ p\in D^m_r$

Sottovarietà riemanniane

$(N, g^N) \subset (M, g^M)$ sottovarietà riemanniana, $\dim N < \dim M$

$(F_1, \ldots, F_m)$ riferimento locale adattato lungo N
$\overset{\text{def}}{\Longleftrightarrow}$ $\left.\begin{array}{l}\langle F_1(p), \ldots, F_n(p)\rangle = T_pN \\ \langle F_{n+1}(p), \ldots, F_m(p)\rangle = T_pN^\perp\end{array}\right\} \forall\, p \in A \subset N$

$(x^1, \ldots, x^m)$ coordinate locali adattate su $A \subset M$
$\overset{\text{def}}{\Longleftrightarrow}$ $(\partial/\partial x^1, \ldots, \partial/\partial x^m)_{|A\cap N}$ riferimento locale adattato lungo N

Nota: $(x^1, \ldots, x^m)$ coordinate locali adattate in senso diff.
$\rightsquigarrow$ $(\partial/\partial x^1, \ldots, \partial/\partial x^m)_{|N}$ riferimento locale lungo N
$\rightsquigarrow$ $(F_1, \ldots, F_m)$ riferimento locale adattato lungo N (G.-S.)
$\rightsquigarrow$ $(x^1, \ldots, x^m)$ coord. locali adattate (mediante $\sqcup_p e_{p|T_pN^\perp}$)

U campo di vettori normali lungo N ($U_p \in T_pN^\perp \subset T_pM\ \forall\, p \in N$)
$\rightsquigarrow$ $L_U : \mathcal{V}N \to \mathcal{V}N$ applicazione $C^\infty(N)$-lineare tale che
$L_U(V) \overset{\text{def}}{=\!=} -\pi(\nabla_V U)$ con $\pi = \sqcup_{p\in N}\pi_p$ proiezione ortogonale

Note: 1) $L_U(V)$ $C^\infty(N)$-lineare rispetto a U
$(L_{fU}(V) = -\pi\nabla_V(fU) = -\pi((Vf)U + f\,\nabla_V U)$
$= -\pi(f\,\nabla_V U) = fL_U(V)\ \forall\, f \in C^\infty(N))$

2) $(F_1, \ldots, F_m)$ riferimento locale adattato
$U = \sum_{i=n+1}^m u^i F_i\,,\ V = \sum_{j=1}^n v^j F_j$
$\Rightarrow L_U(V) = \sum_{i,j} u^i v^j L_{F_i}(F_j)$ dove
$L_{F_i}F_j$ dipende solo dal riferimento

3) $L_U(V)_p$ dipende solo da V_p e $U_p\ \forall\, p \in N$

$(N, g^N) \subset (M, g^M)$ sottovarietà riemanniana, $p \in N$
$\rightsquigarrow$ $L_p : T_pN^\perp \to \operatorname{End} T_pN$ applicazione t.c. $L_p(u) = L_u$ definita
$L_u(v) \overset{\text{def}}{=\!=} L_U(V)_p\ \forall\, U, V$ come sopra con $U_p = u$ e $V_p = v$
$\hookleftarrow$ operatore forma associato al vettore normale u
$\rightsquigarrow$ $L_p : T_pN^\perp \to \operatorname{Bil} T_pN$ applicazione t.c. $L_u(v, w) \overset{\text{def}}{=\!=} g_p(L_u(v), w)$

Nota: entrambe le applicazioni sono ben definite e lineari
(non dipendono dalla scelta delle estensioni U e V)

Prop. $(N, g^N) \subset (M, g^M)$ sottovarietà riemanniana, $u \in T_pN^\perp$
$\Rightarrow L_u(\cdot)$ operatore simm. e $L_u(\cdot,\cdot)$ forma bilineare simm.

Dim. $V \in \mathcal{V}N \Rightarrow g(U,V) = 0 \Rightarrow g(\nabla_W U, V) + g(U, \nabla_W V) = 0$
$W \in \mathcal{V}N \Rightarrow g(U,W) = 0 \Rightarrow g(\nabla_V U, W) + g(U, \nabla_V W) = 0$
$\Rightarrow g(L_U V, W) - g(L_U W, V) = g(U, [V,W]) = 0$

Note: 1) $C \subset M$ curva diff. regolare orientata $\rightsquigarrow$ riferim. di Frenet
(riferm. $(F_1, \ldots, F_m)$ lungo C def. $F_1 = T$ e $F_k = \nabla_T F_{k-1}$
$\rightsquigarrow$ riferim. ortonormale lungo C ottenuto con G.-S.)
2) $N \subset M$ orientate e $n = m-1 \rightsquigarrow L_p \stackrel{\text{def}}{=\!=} L_u$ con
$u =$ unico versore normale positivo rispetto alle orient.
$\rightsquigarrow$ direzioni principali e forma normale (teor. spettrale)
curv. principali, curv. media, curv. di Gauss-Kronecker
(oper. forma e II forma fond. per superf. orient. in R^3)

Prop. $(N, g^N) \subset (M, g^M)$ sottovarietà riemanniana
$\Rightarrow L_u(v,w) = g_p(\nabla^M_v W - \nabla^N_v W, u)$
$\forall\, u \in T_pN^\perp$, $v \in T_pN$, $W \in \mathcal{V}N$ con $W_p = w$

Dim. U campo di vettori normali lungo N tale che $U_p = u$
$\rightsquigarrow g_p(\nabla^M_v W - \nabla^N_v W, u)$
$= g_p(\nabla^M_v W, u) = v\, g(W,U) - g_p(w, \nabla^M_v U)$
$= -\, g_p(w, \nabla^M_v U) = g_p(w, L_u(v)) = L_u(v,w)$

$(N, g^N) \subset (M, g^M)$ sottovarietà riemanniana, $p \in N$
$\rightsquigarrow L_p : T_pN \times T_pN \to T_pN^\perp$ applicazione tale che
$L_p(v,w) \stackrel{\text{def}}{=\!=} \nabla^M_v W - \nabla^N_v W \quad \forall\, W \in \mathcal{V}N$ con $W_p = w$

Note: 1) L_p è ben definita (non dipende dalla scelta di W)
bilineare e simmetrica (proposizione precedente)
2) $(F_1, \ldots, F_m)$ riferimento locale adattato
$v = \sum_{i=1}^n v^i F_i$, $w = \sum_{j=1}^n w^j F_j$
$\Rightarrow L_p(v,w) = \sum_{i,j} v^i w^j L_p(F_i, F_j)$
3) $L_{ij} \stackrel{\text{def}}{=\!=} L(F_i, F_j)$ campo di vettori normali lungo N
che dipende solo dal riferimento

$\rightsquigarrow L_{ij} = \sum_{k=n+1}^m \ell_{ij}^k F_k \,,\, \ell_{ijk} = g(L_{ij}, F_k)$

t.c. $\ell_{ijk} = \sum_{h=n+1}^m g_{kh}\ell_{ij}^h \,,\, \ell_{ij}^k = \sum_{h=n+1}^m g^{kh}\ell_{ijh}$

4) $(\ell_{ijk})_{i,j}$ matrice di $L_{F_k}(\cdot,\cdot)$ rispetto al rif. $(F_1,\dots,F_n)$

Corol. $(N, g^N) \subset (M, g^M)$ sottovarietà geodetica in $p \in N$

$\Leftrightarrow L_u(v) = 0 \;\; \forall u \in T_pN^\perp \,,\, v \in T_pN$

$\Leftrightarrow L_u(v,w) = 0 \;\; \forall u \in T_pN^\perp \,,\, v,w \in T_pN$

$\Leftrightarrow L_p(v,w) = 0 \;\; \forall v,w \in T_pN$

$\Leftrightarrow \nabla_v^N W = \nabla_v^M W \;\; \forall v \in T_pN \,,\, W \in \mathcal{V}N$

Dim. N geodetica in $p \in N \Leftrightarrow \gamma_v^N = \gamma_{v|}^M \;\; \forall v \in T_pN$

$\Leftrightarrow \nabla_v^N V = \nabla_v^M V = 0$ con $V = d\gamma_v^N(t)/dt$ campo vett. su C_v^N

$\Leftrightarrow L_u(v,v) = 0 \;\; \forall u \in T_pN^\perp \,,\, v \in T_pN$ (proposizione sopra)

$\Leftrightarrow L_u(v,w) = 0 \;\; \forall u \in T_pN^\perp \,,\, v,w \in T_pN$ (simmetria)

(che equivale a $L_u(v) = 0 \;\; \forall u \in T_pN^\perp \,,\, v \in T_pN$)

$\Leftrightarrow L_p(v,w) = 0 \;\; \forall v,w \in T_pN$ (arbitrarietà di u)

$\Leftrightarrow \nabla_v^N W = \nabla_v^M W \;\; \forall v \in T_pN \,,\, W \in \mathcal{V}N$ (definizione)

Prop. $(N, g^N) \subset (M, g^M)$ sottovarietà riemanniana

tale che $\dim M > \dim N \geq 2$, $\sigma = \langle v_1, v_2\rangle \subset T_pN \Rightarrow$

$$K^N(\sigma) = K^M(\sigma) + \frac{g_p(L_p(v_1,v_1), L_p(v_2,v_2)) - \|L_p(v_1,v_2)\|_p^2}{g_p(v_1,v_1)\, g_p(v_2,v_2) - g_p(v_1,v_2)^2}$$

Dim. $V_1, V_2 \in \mathcal{V}N$ tali che $V_1(p) = v_1$ e $V_2(p) = v_2 \rightsquigarrow$

$$\begin{aligned}
&(K^N(\sigma) - K^M(\sigma))(g_p(v_1,v_1)\, g_p(v_2,v_2) - g_p(v_1,v_2)^2)\\
&\quad = R^M(v_1,v_2,v_1,v_2) - R^N(v_1,v_2,v_1,v_2)\\
&\quad = g_p(\nabla^M_{V_1}\nabla^M_{V_2}V_1 - \nabla^M_{V_2}\nabla^M_{V_1}V_1 - \nabla^M_{[V_1,V_2]}V_1, V_2)\\
&\qquad - g_p(\nabla^N_{V_1}\nabla^N_{V_2}V_1 - \nabla^N_{V_2}\nabla^N_{V_1}V_1 - \nabla^N_{[V_1,V_2]}V_1, V_2)\\
&\quad = g_p(\nabla^M_{V_1}\nabla^M_{V_2}V_1 - \nabla^M_{V_2}\nabla^M_{V_1}V_1 - \nabla^N_{V_1}\nabla^N_{V_2}V_1 + \nabla^N_{V_2}\nabla^N_{V_1}V_1, V_2)\\
&\quad = g_p(\nabla^M_{V_1}\nabla^M_{V_2}V_1 - \nabla^M_{V_2}\nabla^M_{V_1}V_1 - \nabla^M_{V_1}\nabla^N_{V_2}V_1 + \nabla^M_{V_2}\nabla^N_{V_1}V_1, V_2)\\
&\quad = g_p(\nabla^M_{V_1}(\nabla^M_{V_2}V_1 - \nabla^N_{V_2}V_1), V_2) - g_p(\nabla^M_{V_2}(\nabla^M_{V_1}V_1 - \nabla^N_{V_1}V_1), V_2)\\
&\quad = v_1\, g(L(V_1,V_2), V_2) - g_p(L_p(v_1,v_2), \nabla^M_{v_1}V_2)\\
&\qquad - v_2\, g(L(V_1,V_1), V_2) + g_p(L_p(v_1,v_1), \nabla^M_{v_2}V_2)\\
&\quad = g_p(L_p(v_1,v_1), \nabla^M_{v_2}V_2) - g_p(L_p(v_1,v_2), \nabla^M_{v_1}V_2)
\end{aligned}$$

$$= g_p(L_p(v_1,v_1), \nabla^M_{v_2}V_2 - \nabla^N_{v_2}V_2) - g_p(L_p(v_1,v_2), \nabla^M_{v_1}V_2 - \nabla^N_{v_1}V_2)$$
$$= g_p(L_p(v_1,v_1), L_p(v_2,v_2)) - g_p(L_p(v_1,v_2), L_p(v_1,v_2))$$

Note: 1) $(F_1,\dots,F_m)$ riferim. adattato t.c. $F_1(p)=v_1$ e $F_2(p)=v_2$

$$\rightsquigarrow K^N(\sigma) = K^M(\sigma) + \frac{g_p(L_{11},L_{22}) - \|L_{12}\|^2}{g_{11}g_{22}-g_{12}^2}$$
$$= K^M(\sigma) + \frac{\sum_{i,j=n+1}^m g_{ij}(\ell^i_{11}\ell^j_{22} - \ell^i_{12}\ell^j_{12})}{g_{11}g_{22}-g_{12}^2}$$

2) $N \subset M$ orientate e $n=m-1 \rightsquigarrow \ell_{ij} \overset{\text{def}}{=\!=} \ell^m_{ij}$ con F_m = unico versore normale positivo rispetto alle orient.

$$\rightsquigarrow K^N(\sigma) = K^M(\sigma) + \frac{\ell_{11}\ell_{22}-\ell_{12}^2}{g_{11}g_{22}-g_{12}^2}$$

3) in particolare, N ipersuperficie in R^m

$$\rightsquigarrow K(\sigma) = \frac{\ell_{11}\ell_{22}-\ell_{12}^2}{g_{11}g_{22}-g_{12}^2}$$

Forme di connessione e curvatura

(M,g) varietà riemanniana, $(F_1,\dots,F_m)$ riferimento locale
$(\varphi^1,\dots,\varphi^m)$ forme duali (in senso lineare, cioè $\varphi^i(F_j)=\delta^i_j$)

$\rightsquigarrow \psi^j_i, \psi_{ij}$ 1-forme diff. tali che $\psi^j_i(V) = \varphi^j(\nabla_V F_i)$, $\psi_{ij}(V) = g(\nabla_V F_i, F_j)$ $\forall\, V$ campo vett. $\forall\, i,j = 1,\dots,m$

↳ forme di connessione

ω^j_i, ω_{ij} 2-forme diff. tali che $\omega^j_i(V,W) = \varphi^j(R(V,W)(F_i))$, $\omega_{ij}(V,W) = g(R(V,W)(F_i),F_j)$ $\forall\, V,W$ campi vett. $\forall\, i,j=1,\dots,m$

↳ forme di curvatura

Note: 1) $\left.\begin{array}{l} \psi^j_i = \sum_k g^{jk}\psi_{ik}\,,\ \omega^j_i = \sum_k g^{jk}\omega_{ik} \\ \psi_{ij} = \sum_k g_{jk}\psi^k_i\,,\ \omega_{ij} = \sum_k g_{jk}\omega^k_i \end{array}\right\}\ \forall\, i,j$

2) $\left.\begin{array}{l} \psi^j_i(F_k) = \Gamma^j_{ki}\,,\ \omega^j_i(F_k,F_h) = R^j_{khi} \\ \psi_{ij}(F_k) = \Gamma_{kij}\,,\ \omega_{ij}(F_k,F_h) = R_{ijkh} \end{array}\right\}\ \forall\, i,j,k,h$

3) $\left.\begin{array}{l} \psi^j_i = \sum_k \Gamma^j_{ki}\varphi^k\,,\ \omega^j_i = \sum_{k<h} R^j_{khi}\varphi^k\wedge\varphi^h \\ \psi_{ij} = \sum_k \Gamma_{kij}\varphi^k\,,\ \omega_{ij} = \sum_{k<h} R_{ijkh}\varphi^k\wedge\varphi^h \end{array}\right\}\ \forall\, i,j$

4) $(F_1, \dots, F_m)$ riferimento locale ortonormale

$$\Rightarrow \left.\begin{array}{l} \psi^i_j = \psi_{ji} = -\psi_{ij} = -\psi^j_i \\ \omega^i_j = \omega_{ji} = -\omega_{ij} = -\omega^j_i \end{array}\right\} \forall\, i, j$$

Prop. (M, g) varietà riemanniana, $(F_1, \dots, F_m)$ riferimento locale

$\Rightarrow$ valgono le seguenti equazioni strutturali

$$\left.\begin{array}{l} 1)\ dg_{ij} = \psi_{ij} + \psi_{ji} \\ 2)\ d\varphi^i = \sum_k \varphi^k \wedge \psi^i_k \\ 3)\ d\psi^j_i = \omega^j_i + \sum_k \psi^k_i \wedge \psi^j_k \end{array}\right\} \forall\, i, j = 1, \dots, m$$

Dim. 1) $dg_{ij}(V) = V g(F_i, F_j) = g(\nabla_V F_i, F_j) + g(F_i, \nabla_V F_j)$

$= \psi_{ij}(V) + \psi_{ji}(V)$ $\forall\, V$ campo di vettori

2) $\sum_k \varphi^k \wedge \psi^i_k(V, W)$

$= \sum_k (\varphi^k(V)\psi^i_k(W) - \varphi^k(W)\psi^i_k(V))$

$= \sum_k (v^k \varphi^i(\nabla_W F_k) - w^k \varphi^i(\nabla_V F_k))$

$= \varphi^i(\sum_k (v^k \nabla_W F_k - w^k \nabla_V F_k))$

$= \varphi^i(\sum_k (\nabla_W (v^k F_k) - (W v^k) F_k - \nabla_V (w^k F_k) + (V w^k) F_k))$

$= \varphi^i(\nabla_W V - \nabla_V W) - W v^i + V w^i$

$= V \varphi^i(W) - W \varphi^i(V) - \varphi^i([V, W])$

$= d\varphi^i(V, W)$ $\forall\, V, W$ campi di vettori

3) $\omega^j_i(V, W) = \varphi^j(R(V, W)(F_i))$

$= \varphi^j(\nabla_V \nabla_W F_i - \nabla_W \nabla_V F_i - \nabla_{[V,W]} F_i)$

$= \varphi^j(\sum_k (\nabla_V(\psi^k_i(W) F_k) - \nabla_W(\psi^k_i(V) F_k))) - \psi^j_i([V, W])$

$= \varphi^j(\sum_k ((V \psi^k_i(W)) F_k + \psi^k_i(W) \nabla_V F_k$

$- (W \psi^k_i(V)) F_k - \psi^k_i(V) \nabla_W F_k)) - \psi^j_i([V, W])$

$= V \psi^j_i(W) + \sum_k \psi^k_i(W) \psi^j_k(V)$

$- W \psi^j_i(V) - \sum_k \psi^k_i(V) \psi^j_k(W) - \psi^j_i([V, W])$

$= d\psi^j_i(V, W) - \sum_k \psi^k_i \wedge \psi^j_k(V, W)$

$= (d\psi^j_i - \sum_k \psi^k_i \wedge \psi^j_k)(V, W)$ $\forall\, V, W$ campi di vettori

Varietà a curvatura costante

$M = (M, g)$ varietà riemanniana, $\dim M \geq 2$

a curvatura (sezionale) costante $K \overset{\text{def}}{\iff} K(\sigma) = K$ per ogni $\sigma \subset TM$

Prop. $M = (M, g)$ varietà riemanniana connessa, $\dim M \geq 3$
$K(\sigma) = K(p)\ \forall\, \sigma \subset T_pM\,,\ p \in M \Rightarrow K(p) = K$ costante
(curvatura sezionale isotropa $\Rightarrow$ costante)

Dim. $(F_1, \ldots, F_m)$ riferimento locale ortonormale
$\rightsquigarrow R_{ijkh}(p) = -K(p)(\delta_{ik}\delta_{jh} - \delta_{ih}\delta_{jk}) \quad \forall\, i, j, k, h = 1, \ldots, m$
(stesse simmetrie e stesse curvature sezionali)
$\Rightarrow \omega_{ij} = -K \sum_{k<h}(\delta_{ik}\delta_{jh} - \delta_{ih}\delta_{jk})\varphi^k \wedge \varphi^h = -K\varphi^i \wedge \varphi^j \quad \forall\, i < j$
$\Rightarrow d\psi_i^j = \omega_i^j + \sum_k \psi_i^k \wedge \psi_k^j = -K\varphi^i \wedge \varphi^j + \sum_k \psi_i^k \wedge \psi_k^j \quad \forall\, i < j$
$\Rightarrow -\,dK \wedge \varphi^i \wedge \varphi^j - K d\varphi^i \wedge \varphi^j + K\varphi^i \wedge d\varphi^j + \sum_k (d\psi_i^k \wedge \psi_k^j - \psi_i^k \wedge d\psi_k^j)$
$= -\,dK \wedge \varphi^i \wedge \varphi^j - K \sum_k \varphi^k \wedge \psi_k^i \wedge \varphi^j + K \sum_k \varphi^i \wedge \varphi^k \wedge \psi_k^j$
$+ \sum_k (\omega_i^k \wedge \psi_k^j - \psi_i^k \wedge \omega_k^j) + \sum_{k,h} (\psi_i^h \wedge \psi_h^k \wedge \psi_k^j - \psi_i^k \wedge \psi_k^h \wedge \psi_h^j)$
$= -\,dK \wedge \varphi^i \wedge \varphi^j = 0 \quad \forall\, i < j$
$\Rightarrow dK \wedge \varphi^i \wedge \varphi^j = \sum_h (dK)_h\, \varphi^h \wedge \varphi^i \wedge \varphi^j = 0 \quad \forall\, i < j$
$\Rightarrow (dK)_h = 0 \quad \forall\, h = 1, \ldots, m$
$\Rightarrow dK = 0 \Rightarrow K$ loc. costante $\Rightarrow K$ costante (M connessa)

Spazi modello:
$$M_K^m = \begin{cases} R^m \rightsquigarrow K(\sigma) = K = 0 \\ S^m_{1/\sqrt{K}} \rightsquigarrow K(\sigma) = K > 0 \\ H^m_{1/\sqrt{-K}} \cong D^m_{1/\sqrt{-K}} \rightsquigarrow K(\sigma) = K < 0 \end{cases}$$

Teorema di Cartan (locale)
$M = (M, g)$ varietà riemanniana, $\dim M = m \geq 2$
M a curvatura costante $K \Leftrightarrow M$ localmente isometrica a M_K^m
$(A_p = B(p, \varepsilon_p) \subset M$ isometrico a $B(p_0, \varepsilon_p) \subset M_K^m\ \forall\, p \in M)$

Dim. $(x^1, \ldots, x^m)$ coordinate normali su $A_p \subset M$
$\rightsquigarrow V = \rho \dfrac{\partial}{\partial \rho} = \sum_i x^i \dfrac{\partial}{\partial x^i}$ campo di vettori radiale
tale che $\nabla_V V = V$ e $\left[V, \dfrac{\partial}{\partial x^j}\right] = -\dfrac{\partial}{\partial x^j} \ \ \forall\, j = 1, \ldots, m$
$\rightsquigarrow (F_1, \ldots, F_m)$ riferimento locale ortonormale
tale che $\nabla_V F_i = 0$ e $F_i(p) = \partial/\partial x^i$
$\rightsquigarrow (\varphi^1, \ldots, \varphi^m)$ forme duali $\rightsquigarrow \psi_i^j = \psi_{ij}\,,\ \omega_i^j = \omega_{ij}$

$$\left.\begin{aligned}\Rightarrow \varphi^k(V) &= x^k \\ \psi_j^k(V) &= \varphi^k(\nabla_V F_j) = 0\end{aligned}\right\} \forall\, j,k = 1,\ldots,m$$

$$(V\varphi^k(V) = Vg(V,F_k) = g(\nabla_V V, F_k) + g(V, \nabla_V F_k) = \varphi^k(V)$$

$$\Rightarrow \varphi^k(V) \text{ omog. di grado 1 t.c. } \partial\varphi^k(V)/\partial x^i = \delta_i^k \text{ in } 0)$$

$a_i^k = \varphi^k(\partial/\partial x^i)$ con $i,j = 1,\ldots,m$

$b_{ij}^k = \psi_j^k(\partial/\partial x^i)$ con $i,j,k = 1,\ldots,m$

$$\begin{aligned}\Rightarrow Va_i^k &= d\varphi^k\Big(V, \frac{\partial}{\partial x^i}\Big) + \frac{\partial}{\partial x^i}\varphi^k(V) + \varphi^k\Big(\Big[V, \frac{\partial}{\partial x^i}\Big]\Big) \\ &= \sum_h (\varphi^h \wedge \psi_h^k)\Big(V, \frac{\partial}{\partial x^i}\Big) + \frac{\partial x^k}{\partial x^i} - \varphi^k\Big(\frac{\partial}{\partial x^i}\Big) \\ &= \sum_h \Big(\varphi^h(V)\psi_h^k\Big(\frac{\partial}{\partial x^i}\Big) - \varphi^h\Big(\frac{\partial}{\partial x^i}\Big)\psi_h^k(V)\Big) + \delta_i^k - a_i^k \\ &= \delta_i^k - a_i^k + \sum_h x^h b_{ih}^k\end{aligned}$$

$$\begin{aligned}Vb_{ij}^k &= d\psi_j^k\Big(V, \frac{\partial}{\partial x^i}\Big) + \frac{\partial}{\partial x^i}\psi_j^k(V) + \psi_j^k\Big(\Big[V, \frac{\partial}{\partial x^i}\Big]\Big) \\ &= \omega_j^k\Big(V, \frac{\partial}{\partial x^i}\Big) + \sum_h (\psi_j^h \wedge \psi_h^k)\Big(V, \frac{\partial}{\partial x^i}\Big) \\ &\quad + \frac{\partial}{\partial x^i}\psi_j^k(V) - \psi_j^k\Big(\frac{\partial}{\partial x^i}\Big) \\ &= \omega_{jk}\Big(\sum_h x^h F_h, \sum_\ell a_i^\ell F_\ell\Big) - b_{ij}^k \\ &= -b_{ij}^k + \sum_{h,\ell} R_{jkh\ell} x^h a_i^\ell \\ &= -b_{ij}^k - \sum_{h,\ell} K(\delta_{jh}\delta_{k\ell} - \delta_{j\ell}\delta_{kh}) x^h a_i^\ell \\ &= -b_{ij}^k - K(x^j a_i^k - x^k a_i^j)\end{aligned}$$

$v \in T_pM$ versore $\rightsquigarrow x = \rho v$ raggio uscente da p con $\rho < \varepsilon_p$

$\rightsquigarrow f_i^k(\rho) = \rho\, a_i^k(\rho v)\,,\ f_{ij}^k(\rho) = \rho\, b_{ij}^k(\rho v)$

$\rightsquigarrow$ problema di Cauchy dipendente solo da K

$$\begin{cases} \dfrac{df_i^k}{d\rho} = a_i^k(\rho v) + Va_i^k(\rho v) = \delta_i^k + \sum_h v^h f_{ih}^k \\ \dfrac{df_{ij}^k}{d\rho} = b_{ij}^k(\rho v) + Vb_{ij}^k(\rho v) = -K(v^j f_i^k - v^k f_i^j) \\ f_i^k(0) = f_{ij}^k(0) = 0 \end{cases}$$

$\Rightarrow f_i^k$, f_{ij}^k a_i^k , b_{ij}^k , $ds_g^2 = \sum_k (\varphi^k)^2 = \sum_{i,j,k} a_i^k a_j^k dx^i dx^j$
univocamente determinate da K in A_p
$\rightsquigarrow f : B(p_0, \varepsilon_p) \to A_p$ con $B(p_0, \varepsilon_p) \subset M_K^m$ isometria
(con $T_{p_0} f : T_{p_0} M_K^m \to T_p M$ isometria arbitraria)
definita dall'identità in coordinate normali
($\varepsilon_p \leq \varepsilon_{p_0}$ con $\varepsilon_{p_0} = \infty$ se $K \leq 0$ e $\varepsilon_{p_0} = \pi/\sqrt{K}$ se $K > 0$)

Note: 1) M_K^m varietà riem. omogenea isotropa per ogni K e $m \geq 2$
($\mathrm{Isom}_p\, M_K^m \cong \mathrm{O}(m)$ agisce trans. sui riferim. ortonorm.)
2) M a curv. costante $\Rightarrow$ localmente omogenea e isotropa
($\mathrm{Isom}_p\, B(p, \varepsilon) \cong \mathrm{O}(m)$ agisce trans. sui rifer. ortonorm.)

Teorema di Cartan (globale)
$M = (M, g)$ varietà riemanniana, $\dim M = m \geq 2$
M sempl. connessa completa a curv. costante $K \Leftrightarrow M \cong M_K^m$

Dim. M completa a curvatura costante K
$\rightsquigarrow f_{p_0} : A_{p_0} \to M$ isometria locale con
$A_{p_0} = M_K^m$ se $K \leq 0$ e $A_{p_0} = M_K^m - \{-p_0\}$ se $K > 0$
$\rightsquigarrow f : M_K^m \to M$ isometria locale
($f = f_{p_1} \cup f_{p_2}$ con p_1 e p_2 non antipodali se $K > 0$)
M connessa (e M_K^m completa) $\Rightarrow f$ rivestimento
M semplicemente connessa $\Rightarrow f$ isometria

Corol. $M = (M, g)$ varietà riemanniana, $\dim M = m \geq 2$
M connessa completa a curvatura costante K
$\Leftrightarrow M \cong M_K^m / G$ con $G \subset \mathrm{Isom}\, M_K^m$ prop. discontinuo

Dim. $\Rightarrow$) $f : \widetilde{M} \to M \cong \widetilde{M}/G_f$ rivestimento universale
$\rightsquigarrow \widetilde{M} = (\widetilde{M}, \widetilde{g})$ semplicemente connessa
tale che f isometria locale e $G_f \subset \mathrm{Isom}\, \widetilde{M}$
$\Rightarrow \widetilde{M}$ completa a curvatura costante $K \Rightarrow \widetilde{M} \cong M_K^m$
$\Leftarrow$) $M \cong M_K^m / G$ con $G \subset \mathrm{Isom}\, M_K^m$
$\Rightarrow \pi : M_K^m \to M$ rivestimento localmente isometrico
$\Rightarrow M$ connessa completa a curvatura costante K