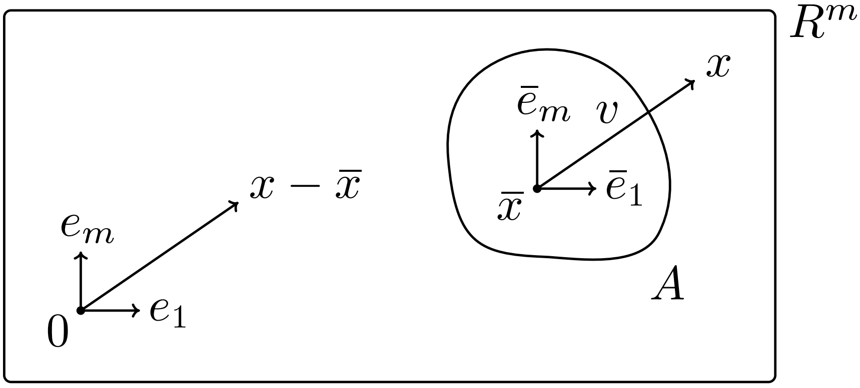
<u>Premessa</u>

$A \subset R^m$ aperto, $\bar{x} \in A \rightsquigarrow T_{\bar{x}}A \stackrel{\text{def}}{=\!=} \{v = \overrightarrow{\bar{x}x} \mid x \in R^m\}$

$\hookrightarrow$ <u>spazio tangente</u> ad A in $\bar{x}$

<u>Note</u>: 1) $T_{\bar{x}}A = T_{\bar{x}}R^m \cong R^m$ $(v \leftrightarrow x - \bar{x})$ spazio vett. di dim m

2) $\{\bar{e}_1, \ldots, \bar{e}_m\} \leftrightarrow \{e_1, \ldots, e_m\}$ base canonica di $T_{\bar{x}}A$



$f : A \to B$ appl. diff. con $A \subset R^m$ e $B \subset R^n$ ap., $\bar{x} \in A$ e $\bar{y} = f(\bar{x})$

$\rightsquigarrow T_{\bar{x}}f : T_{\bar{x}}A \to T_{\bar{y}}B$ <u>applicazione tangente</u> a f in $\bar{x}$

$$\begin{array}{ccccl} v & \mapsto & w & = & T_{\bar{x}}f(v) \\ \updownarrow & & \updownarrow & & \\ x - \bar{x} & \mapsto & y - \bar{y} & = & d_{\bar{x}}f(x - \bar{x}) = J_{\bar{x}}f \cdot (x - \bar{x}) \end{array}$$

<u>Note</u>: 1) $T_{\bar{x}}f$ lineare, $T_{\bar{x}}\mathrm{id}_A = \mathrm{id}_{T_{\bar{x}}A}$, $T_{\bar{x}}(g \circ f) = T_{\bar{y}}g \circ T_{\bar{x}}f$

2) $T_{\bar{x}}f$ isomorfismo $\Leftrightarrow$ f diffeo locale intorno a $\bar{x}$

3) $J_{\bar{x}}f$ matrice di $T_{\bar{x}}f$ rispetto alle basi canoniche

$g : A \to R$ funzione diff. con $A \subset R^m$ aperto, $\bar{x} \in A$

$v = \Sigma_i v^i \bar{e}_i \in T_{\bar{x}}A \rightsquigarrow v\,g = \dfrac{\partial g}{\partial v} = \sum_i v^i \dfrac{\partial g}{\partial x^i} \in R$

$\hookrightarrow$ <u>derivata di g rispetto a v</u>

<u>Note</u>: 1) vg R-lineare rispetto a $v \in T_{\bar{x}}A$

2) $v\,(ag + bh) = a\,(v\,g) + b\,(v\,h)$ $\forall\, a, b \in R$ $\forall\, g, h : A \to R$ diff.

3) $v\,(g \cdot h) = (v\,g) \cdot h + g \cdot (v\,h)$ $\forall\, g, h : A \to R$ diff.

$v : C^\infty(A) \to R$ <u>derivazione</u> (= R-lineare + regola di Leibnitz)

$$v = \frac{\partial}{\partial v} = \sum_i v^i \frac{\partial}{\partial x^i} \quad \left(\{\bar{e}_1, \ldots, \bar{e}_m\} \rightsquigarrow \left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\right\}\right)$$

<u>Note</u>: 1) $v\,g \leftrightarrow T_{\bar{x}}g(v)$ (mediante l'isomorf. canonico $T_{g(\bar{x})}R \cong R$)

2) $f : A \to B$ applicazione diff., $g : B \to R$ funzione diff.

$v = \sum_i v^i \dfrac{\partial}{\partial x^i} \in T_{\bar{x}}A,\ w = \sum_j w^j \dfrac{\partial}{\partial y^j} = T_{\bar{x}}f(v) \in T_{\bar{y}}B$

$\Rightarrow\ w\,g = T_{\bar{y}}g(w) = T_{\bar{y}}g(T_{\bar{x}}f(v)) = T_{\bar{x}}(g \circ f)(v) = v\,(g \circ f)$

$\left(\text{in coord. } \sum_j w^j \dfrac{\partial g}{\partial y^j} = \sum_{i,j} v^i \dfrac{\partial y^j}{\partial x^i}\dfrac{\partial g}{\partial y^j} = \sum_i v^i \dfrac{\partial (g \circ f)}{\partial x^i}\right)$

$A \subset R^m$ aperto, $\bar{x} \in A$

$\rightsquigarrow T^*_{\bar{x}}A \overset{\text{def}}{=\!=} (T_{\bar{x}}A)^* = \{l : T_{\bar{x}}A \to R \mid l \text{ lineare}\}$

<u>spazio cotangente</u> ad A in $\bar{x}$

<u>Note</u>: 1) $T^*_{\bar{x}}A = T^*_{\bar{x}}R^m \cong R^m$

2) $\{\bar{e}_1, \ldots, \bar{e}_m\} \rightsquigarrow \{\bar{e}^1, \ldots, \bar{e}^m\}$ base duale $(\bar{e}^i(\bar{e}_j) = \delta^i_j)$

$f : A \to B$ appl. diff. con $A \subset R^m$ e $B \subset R^n$ ap., $\bar{x} \in A$ e $\bar{y} = f(\bar{x})$

$\rightsquigarrow T^*_{\bar{x}}f = (T_{\bar{x}}f)^* : T^*_{\bar{y}}B \to T^*_{\bar{x}}A$ (definita $l \mapsto l \circ T_{\bar{x}}f$)

<u>applicazione cotangente</u> a f in $\bar{x}$

<u>Note</u>: 1) $T^*_{\bar{x}}f$ lineare, $T^*_{\bar{x}}\mathrm{id}_A = \mathrm{id}_{T^*_{\bar{x}}A}$, $T^*_{\bar{x}}(g \circ f) = T^*_{\bar{x}}f \circ T^*_{\bar{y}}g$

2) $T^*_{\bar{x}}f$ isomorfismo $\Leftrightarrow$ f diffeo locale intorno a $\bar{x}$

3) $(J_{\bar{x}}f)^*$ matrice di $T^*_{\bar{x}}f$ rispetto alle basi canoniche duali

$g : A \to R$ funzione diff. con $A \subset R^m$ aperto, $\bar{x} \in A$

$\rightsquigarrow dg : T_{\bar{x}}A \to R$ definita $v \mapsto dg(v) \overset{\text{def}}{=\!=} v\,g$

<u>differenziale</u> di g in $\bar{x}$

<u>Note</u>: 1) $dg \in T^*_{\bar{x}}A$ (dg è lineare)

2) $x^i : A \to R$ funzioni coord. $\Rightarrow dx^i\left(\dfrac{\partial}{\partial x^j}\right) = \dfrac{\partial x^i}{\partial x^j} = \delta^i_j$

$\rightsquigarrow \{dx^1, \ldots, dx^m\}$ base duale di $T^*_{\bar{x}}A$

3) $dg = \sum_i \dfrac{\partial g}{\partial x^i}dx^i$ (= differenziale totale di g)

4) $dg \leftrightarrow d_{\bar{x}}g \leftrightarrow T_{\bar{x}}g$ (mediante gli isomorf. canonici
$T_{\bar{x}}R^m \cong R^m$ e $T_{g(\bar{x})}R \cong R$)

Spazi tangenti

M m-varietà differenziabile, $p \in M$

$\Gamma_p M \stackrel{\text{def}}{=\!=} \{\gamma : I \to M \text{ differenziabile} \mid 0 \in I \subset R\,,\ \gamma(0) = p\}$

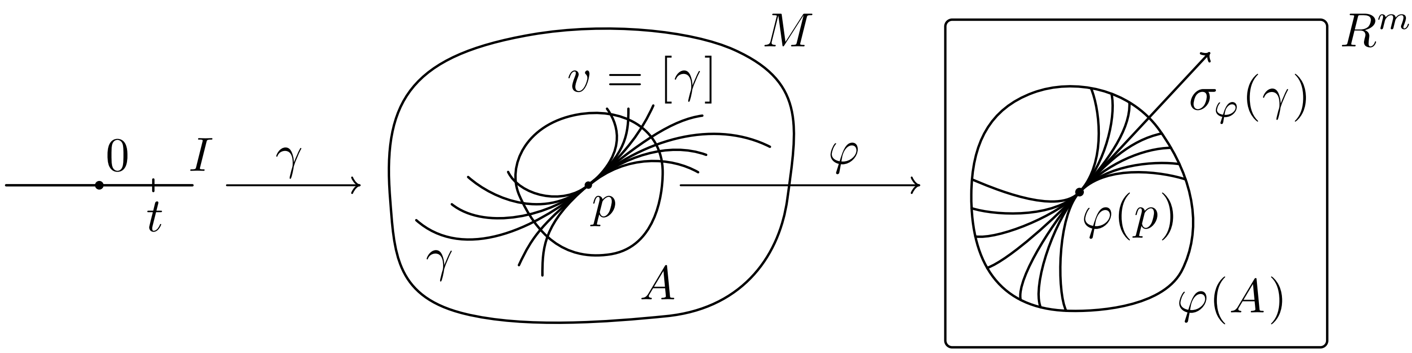
curve differenziabili in M uscenti da p

(A, φ) carta locale di M con $p \in A$

$$\rightsquigarrow \sigma_\varphi : \Gamma_p M \to T_{\varphi(p)} R^m \cong R^m$$
$$\gamma \mapsto \sigma_\varphi(\gamma) = (\varphi \circ \gamma)'(0) = T_0(\varphi \circ \gamma)\left(\frac{\partial}{\partial t}\right)$$

$T_p M \stackrel{\text{def}}{=\!=} \Gamma_p M / \sim_\varphi$ spazio tangente a M in p

$v = [\gamma] \in T_p M$ vettore tangente a M in p (velocità iniziale di γ)



Note: 1) $\sigma_\varphi \rightsquigarrow \tau_\varphi = \sigma_\varphi / \sim_\varphi : T_p M \to T_{\varphi(p)} R^m$ biiettiva

$\rightsquigarrow T_p M$ spazio vettoriale reale t.c. $\tau_\varphi : T_p M \cong T_{\varphi(p)} R^m$

2) $T_p M$ è ben definito, cioè non dipende dalla carta (A, φ)

$((B, \psi)$ carta con $p \in B \Rightarrow \sigma_\psi(\gamma) = T_{\varphi(p)}(\psi \circ \varphi^{-1})(\sigma_\varphi(\gamma)))$

$$\begin{tikzcd} & \Gamma_p M \arrow[dl, "\sigma_\varphi"'] \arrow[dr, "\sigma_\psi"] & \\ T_{\varphi(p)} R^m \arrow[rr, "\cong", "T_{\varphi(p)}(\psi \circ \varphi^{-1})"'] & & T_{\psi(p)} R^m \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & T_p M \arrow[dl, "\tau_\varphi"', "\cong"] \arrow[dr, "\tau_\psi", "\cong"'] & \\ T_{\varphi(p)} R^m \arrow[rr, "\cong", "T_{\varphi(p)}(\psi \circ \varphi^{-1})"'] & & T_{\psi(p)} R^m \end{tikzcd}$$

3) (A, φ) con coord. $(x^1, \ldots, x^m) \rightsquigarrow T_p M = \left\langle \dfrac{\partial}{\partial x^1}, \ldots, \dfrac{\partial}{\partial x^m} \right\rangle$

(B, ψ) con coord. $(y^1, \ldots, y^m) \rightsquigarrow T_p M = \left\langle \dfrac{\partial}{\partial y^1}, \ldots, \dfrac{\partial}{\partial y^m} \right\rangle$

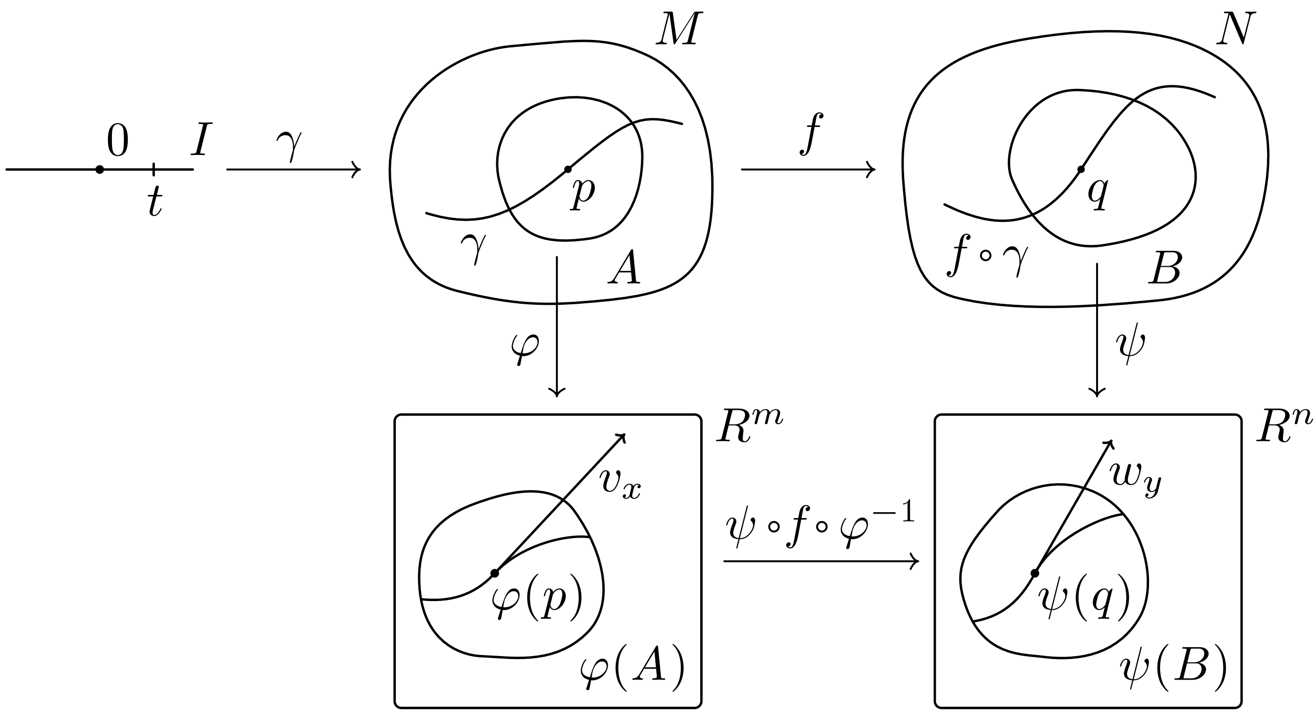
$$v = [\gamma] \xrightarrow{\tau_\varphi} (\varphi \circ \gamma)'(0) = \sum_i v_x^i \frac{\partial}{\partial x^i} \text{ con } v_x^i = \frac{\partial x^i}{\partial t}$$

$$v = [\gamma] \xrightarrow{\tau_\psi} (\psi \circ \gamma)'(0) = \sum_j v_y^j \frac{\partial}{\partial y^j} \text{ con } v_y^j = \frac{\partial y^j}{\partial t}$$

$$\rightsquigarrow v_y^j = \sum_i \frac{\partial y^j}{\partial x^i} v_x^i \quad \left(\frac{\partial}{\partial y^j} = \sum_i \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i} \right)$$

4) $M = A \subset R^m$ ap., $p = \overline{x} \Rightarrow T_p M = T_{\overline{x}} A$ $([\gamma] \leftrightarrow \gamma'(0))$

$f : M \to N$ appl. diff. con M e N varietà diff., $p \in M$, $q = f(p)$
$\rightsquigarrow T_p f : T_p M \to T_q N$ definita $v = [\gamma] \mapsto w = T_p f(v) \overset{\text{def}}{=\!=} [f \circ \gamma]$
↳ <u>applicazione tangente</u> a f in p



<u>Note</u>: 1) $T_p f$ è ben definita e lineare $(w_y = T_{\varphi(p)}(\psi \circ f \circ \varphi^{-1})(v_x))$

2) $T_p \mathrm{id}_M = \mathrm{id}_{T_p M}$, $T_p(g \circ f) = T_{f(p)} g \circ T_p f$, $\mathrm{rg}_p f = \mathrm{rg}\, T_p f$

3) $T_p f$ isomorfismo $\Leftrightarrow$ f diffeo locale intorno a p

4) (A, φ) con coord. $(x^1, \ldots, x^m) \rightsquigarrow T_p M = \langle \partial/\partial x^1, \ldots, \partial/\partial x^m \rangle$
(B, ψ) con coord. $(y^1, \ldots, y^m) \rightsquigarrow T_q N = \langle \partial/\partial y^1, \ldots, \partial/\partial y^m \rangle$
$w_y^j = \sum_i \frac{\partial y^j}{\partial x^i} v_x^i$ $(J_{\varphi(p)}(\psi \circ f \circ \varphi^{-1})$ matrice di $T_p f$ rispetto alle basi canoniche indotte da φ e $\psi)$

5) (A, φ) carta locale di M intorno a $p \Rightarrow T_p \varphi = \tau_\varphi$

6) $M = A \subset R^m$ e $N = B \subset R^n$ aperti, $p = \overline{x} \Rightarrow T_p f = T_{\overline{x}} f$

M varietà differenziabile, $v = [\gamma] \in T_pM$

$\rightsquigarrow v : C^\infty(M) \to R$ definita $g \mapsto v\,g = \dfrac{\partial g}{\partial v} = (g \circ \gamma)'(0)$

derivazione rispetto a v (R-lineare + regola di Leibnitz)

Note: 1) $v\,g \leftrightarrow T_pg(v)$ (mediante l'isomorf. canonico $T_{f(p)}R \cong R$)

2) $f : M \to N$ appl. diff., $g \in C^\infty(N) \Rightarrow T_f(v)\,g = v\,(g \circ f)$

3) in coord. locali $(x^1, \dots, x^m)$ si ha: $v\,g = \sum_i v_x^i \dfrac{\partial g}{\partial x^i}$

M m-varietà diff., $N \subset M$ n-sottovarietà diff., $p \in N$

$\Rightarrow T_pN \subset T_pM$ sottospazio vettoriale $([\gamma]_N \in T_pN \rightsquigarrow [\gamma]_M \in T_pM)$

Note: 1) $i : N \hookrightarrow M$ inclusione $\rightsquigarrow T_pi : T_pN \hookrightarrow T_pM$ inclusione

2) (A, φ) carta adattata $\Rightarrow T_pN = \left\langle \dfrac{\partial}{\partial x^1}, \dots, \dfrac{\partial}{\partial x^n} \right\rangle \subset T_pM$

Prop. M m-varietà diff., $N \subset M$ n-sottovarietà diff., $p \in N$

1) $E : A \to R^{m-n}$ equaz. loc. reg. di N in A int. ap. di p

$\Rightarrow T_pN = \operatorname{Ker} T_pE \subset T_pM$ ($T_pE = 0$ equaz. di T_pN)

2) $f : D \to A$ par. loc. reg. di N, $A \subset N$ int. ap. di $p = f(x)$

$\Rightarrow T_pN = \operatorname{Im} T_xf \subset T_pM$ (T_xf param. di T_pN)

Dim. 1) $T_pN \subset \operatorname{Ker} T_pE$ $([\gamma] \in T_pN \Rightarrow E \circ \gamma = 0 \Rightarrow T_pE([\gamma]) = 0)$

E regolare $\Rightarrow \dim \operatorname{Ker} T_pE = n \Rightarrow T_pN = \operatorname{Ker} T_pE$

2) $\operatorname{Im} T_xf \subset T_pN$ $([\gamma] \in T_xD \Rightarrow f \circ \gamma \subset N \Rightarrow T_xf([\gamma]) \in T_pN)$

f regolare $\Rightarrow \dim \operatorname{Im} T_xf = n \Rightarrow \operatorname{Im} T_f = T_pN$

Nota: $N \subset R^m$ n-sottovarietà differenziabile, $\bar{x} \in N$

$T_{\bar{x}}N \subset T_{\bar{x}}R^m$ sottospazio affine di R^m $(v = \overrightarrow{\bar{x}x} \leftrightarrow x)$

$E(x) = 0$ eq. loc. reg. $\rightsquigarrow T_{\bar{x}}E(x - \bar{x}) = 0$ equaz. cartesiana

$x = f(t)$ par. loc. reg. $\rightsquigarrow x - \bar{x} = T_{\bar{t}}f(t - \bar{t})$ equaz. param.

Esempi: 1) C curva diff. regolare in R^2, $\bar{p} = (\bar{x}, \bar{y}) \in C$

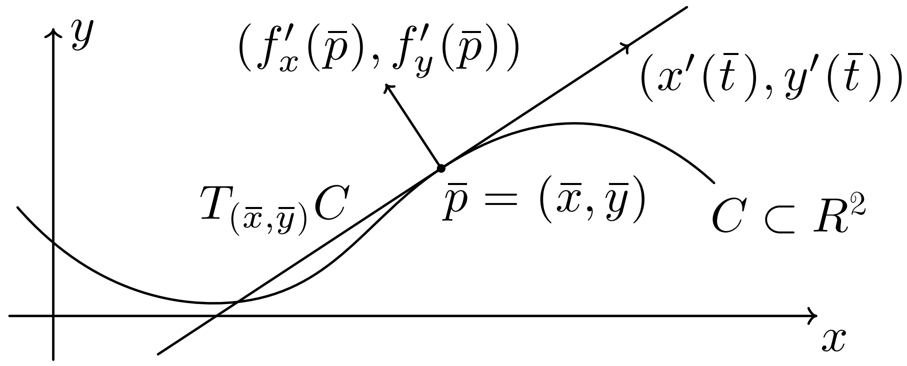
$f(x, y) = 0$ equazione loc. reg. di C intorno a $\bar{p}$

$\rightsquigarrow f'_x(\bar{p})(x - \bar{x}) + f'_y(\bar{p})(y - \bar{y}) = 0$

equazione cartesiana di $T_{\bar{p}}C \subset R^2$

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad \text{param. loc. reg. di } C \text{ intorno a } (\bar{x}, \bar{y})$$

$$\rightsquigarrow \begin{cases} x = \bar{x} + x'(\bar{t}\,)(t - \bar{t}\,) \\ y = \bar{y} + y'(\bar{t}\,)(t - \bar{t}\,) \end{cases} \quad \text{equaz. param. di } T_{\bar{p}}C \subset R^2$$

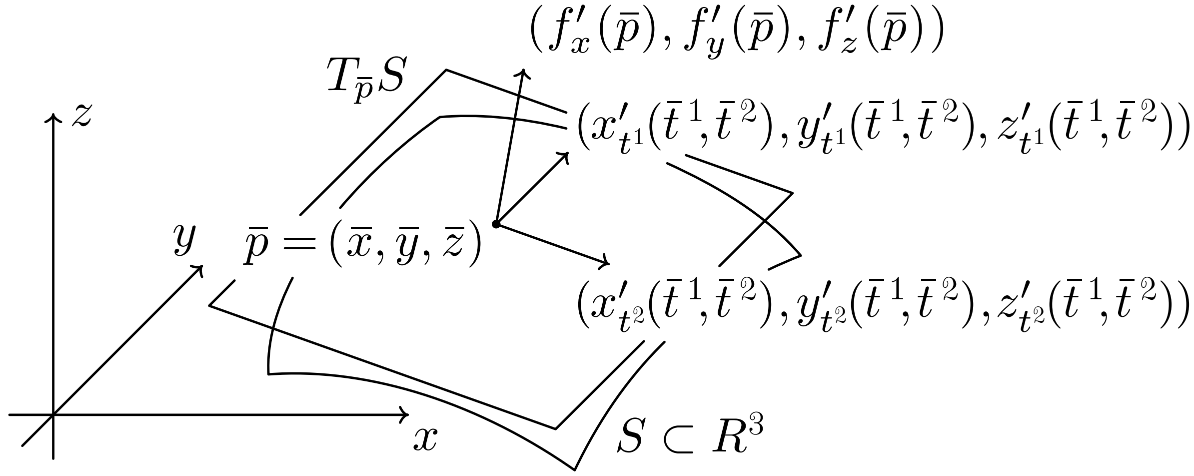


2) S superficie diff. regolare in R^3, $\bar{p} = (\bar{x}, \bar{y}, \bar{z}) \in S$
$f(x, y, z) = 0$ equazione loc. reg. di S intorno a $\bar{p}$
$\rightsquigarrow f'_x(\bar{p})(x - \bar{x}) + f'_y(\bar{p})(y - \bar{y}) + f'_z(\bar{p})(z - \bar{z}) = 0$
equazione cartesiana di $T_{\bar{p}}S \subset R^3$

$$\begin{cases} x = x(t^1, t^2) \\ y = y(t^1, t^2) \\ z = z(t^1, t^2) \end{cases} \quad \text{param. loc. reg. di } S \text{ intorno a } \bar{p}$$

$$\rightsquigarrow \begin{cases} x = \bar{x} + x'_{t^1}(\bar{t}^{\,1}, \bar{t}^{\,2})(t^1 - \bar{t}^{\,1}) + x'_{t^2}(\bar{t}^{\,1}, \bar{t}^{\,2})(t^2 - \bar{t}^{\,2}) \\ y = \bar{y} + y'_{t^1}(\bar{t}^{\,1}, \bar{t}^{\,2})(t^1 - \bar{t}^{\,1}) + y'_{t^2}(\bar{t}^{\,1}, \bar{t}^{\,2})(t^2 - \bar{t}^{\,2}) \\ z = \bar{z} + z'_{t^1}(\bar{t}^{\,1}, \bar{t}^{\,2})(t^1 - \bar{t}^{\,1}) + z'_{t^2}(\bar{t}^{\,1}, \bar{t}^{\,2})(t^2 - \bar{t}^{\,2}) \end{cases}$$

equaz. param. di $T_{\bar{p}}S \subset R^3$



3) C curva diff. regolare in R^3, $\bar{p} = (\bar{x}, \bar{y}, \bar{z}) \in C$

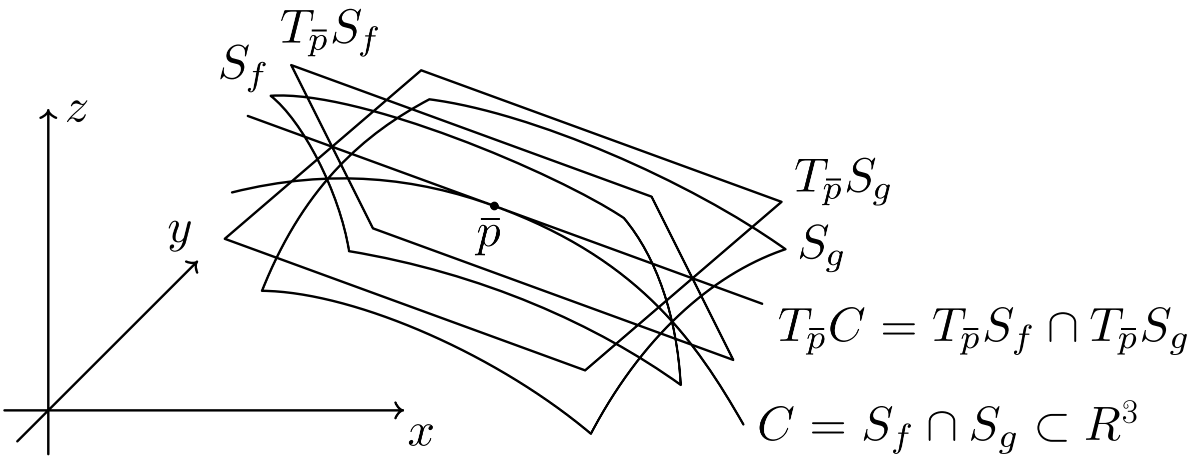
$$\begin{cases} f(x,y,z) = 0 \\ g(x,y,z) = 0 \end{cases} \text{ equaz. loc. reg. di } C \text{ intorno a } (\bar{x}, \bar{y}, \bar{z})$$

$$\rightsquigarrow \begin{cases} f'_x(\bar{p})(x-\bar{x}) + f'_y(\bar{p})(y-\bar{y}) + f'_z(\bar{p})(z-\bar{z}) = 0 \\ f'_x(\bar{p})(x-\bar{x}) + f'_y(\bar{p})(y-\bar{y}) + f'_z(\bar{p})(z-\bar{z}) = 0 \end{cases}$$

equazione cartesiana di $T_{\bar{p}}C \subset R^3$

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \text{ param. loc. reg. di } C \text{ intorno a } \bar{p}$$

$$\rightsquigarrow \begin{cases} x = \bar{x} + x'_t(\bar{t})(t-\bar{t}) \\ y = \bar{y} + y'_t(\bar{t})(t-\bar{t}) \\ z = \bar{z} + z'_t(\bar{t})(t-\bar{t}) \end{cases} \text{ equaz. param. di } T_{\bar{p}}S \subset R^3$$



$f : M \to N$ applicazione differenziabile

<u>L-regolare</u> (o <u>trasversale</u> rispetto a L) con $L \subset N$ sottovar. diff.

$\overset{\text{def}}{\iff} T_pf(T_pM) + T_qL = T_qN$ per ogni $p \in M$ tale che $f(p) = q \in L$

<u>Note</u>: 1) se $m + \ell < n$ allora f è L-regolare $\Leftrightarrow f(M) \cap L = \varnothing$

2) se $m \geq n$ allora $q \in M$ val. reg. di $f \Leftrightarrow f$ è $\{q\}$-regolare

$M, L \subset N$ sottovarietà differenziabili

<u>trasversali</u> $\overset{\text{def}}{\iff}$ l'inclusione $i_M : M \hookrightarrow N$ è L-regolare

$\iff$ l'inclusione $i_L : L \hookrightarrow N$ è M-regolare

$\iff T_pM + T_pL = T_pN$ per ogni $p \in M \cap L$

<u>Nota</u>: se $m + \ell < n$ allora M e L trasversali $\Leftrightarrow M \cap L = \varnothing$

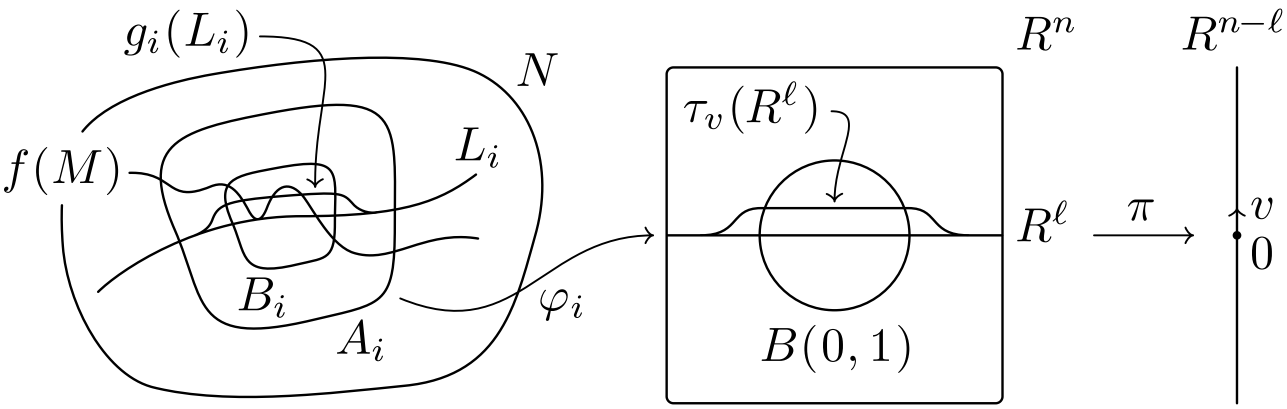
Prop. 1) $f : M \to N$ appl. diff. L-regolare con $L \subset N$ sottovar. diff.
$m + \ell \geq n \Rightarrow f^{-1}(L) \subset M$ $(m+\ell-n)$-sottovar. diff.
2) $M, L \subset N$ sottovarietà differenziabili trasversali
$m + \ell \geq n \Rightarrow M \cap L \subset N$ $(m+\ell-n)$-sottovar. diff.

Dim. 1) $p \in f^{-1}(L) \rightsquigarrow (A, \varphi) \in \mathcal{S}_N$ carta adattata a L in $q = f(p)$
$\rightsquigarrow E = \pi \circ \varphi \circ f : f^{-1}(A) \to R^{n-\ell}$ diff. regolare
$\rightsquigarrow$ equazione loc. regolare per $f^{-1}(L)$ in p
2) segue da 1) con $f = i_M : M \hookrightarrow N$ inclusione

Teorema di approssimazione trasversale

1) $f : M \to N$ appl. diff., $L \subset N$ sottov. diff., $\varepsilon : N \to \,]0, \infty[$ cont.
$\Rightarrow \exists\, \mathrm{id}_N \cong_\varepsilon h \in \mathrm{Diffeo}\, N$ tale che $g = h \circ f$ è L-regolare
$(\exists\, H \colon \mathrm{id}_N \simeq h$ t.c. $h_t \in \mathrm{Diffeo}\, N,\ d(h_t(q), q) < \varepsilon(q)\ \forall\, q \in N)$
2) $M, L \subset N$ sottovarietà diff., $\varepsilon : N \to \,]0, \infty[$ continua
$\Rightarrow \exists\, \mathrm{id}_N \cong_\varepsilon h \in \mathrm{Diffeo}\, N$ t.c. $M' = h(M)$ e L sono trasversali

Dim. 1) $\mathcal{A} = \{(A_i, \varphi_i)\}_{i \geq 1}$ famiglia loc. finita di carte speciali di N
adattate a L tali che $\mathcal{B} = \{B_i = \varphi_i^{-1}(B(0,1))\}_{i \geq 1}$ ricopre L
$\rightsquigarrow H_i : h_{i-1} \cong_{\delta_i} h_i \in \mathrm{Diffeo}\, N$ con $\delta_i(q) = \varepsilon(h_{i-1}^{-1}(q))/2^i$
t.c. $h_i \circ f$ è L_i-regolare con $L_i = L \cap (B_1 \cup \ldots \cup B_i)$
$h_{i|N-A_i} = h_{i-1|N-A_i}$ e $H_{i|N-A_i}$ costante (omot. rel.)



(induz. su $i \geq 0$ a partire da $h_0 = \mathrm{id}_M$ con $A_0 = L_0 = \varnothing$
$v \in R^{n-\ell}$ valore regolare per $\pi \circ \varphi_i \circ f : f^{-1}(A_i) \to R^{n-\ell}$
$\rightsquigarrow \tau_v : R^n \to R^n$ definita $\tau_v(x) = x + \eta(x) v$
$\rightsquigarrow g_i \in \mathrm{Diffeo}\, N$ def. $g_{i|A_i} = \varphi_i^{-1} \circ \tau_v \circ \varphi_i,\ g_{i|M-A_i} = \mathrm{id}$
t.c. $(\varphi_i^{-1} \circ \tau_{tv} \circ \varphi_i)_t : \mathrm{id}_N \cong g_i$ e $h_{i-1} \circ f$ è $g_i(L_i)$-reg.

$\rightsquigarrow h_i = g_i^{-1} \circ h_{i-1} \in \operatorname{Diffeo} N$ con le propr. desiderate
se $\|v\|$ suff. piccolo per preservare L_{i-1}-regolarità)

$\rightsquigarrow H : \mathrm{id}_N \cong_\varepsilon h \in \operatorname{Diffeo} N$ tale che $g = h \circ f$ è L-regolare
($h = \lim_{i\to\infty} h_i$ ben definito poiché $\mathcal{A}$ localmente finita
$H =$ concat. delle H_i riscalate su $[1{-}1/2^{i-1}, 1{-}1/2^i]$
per induzione su $i \geq 0$ si ha $\forall\, t \in [1{-}1/2^{i-1}, 1{-}1/2^i]$

$$\begin{aligned} d(h_t(q), q) &= d((h_i)_{1-1/2^{i-1}+t/2^i}(h_{i-1}(q)), q) \\ &< \delta_i(h_{i-1}(q)) + d(h_{i-1}(q), q) \\ &< \varepsilon(q)/2^i + (1{-}1/2^{i-1})\varepsilon(q) = (1{-}1/2^i)\varepsilon(q)) \end{aligned}$$

2) segue da 1) con $f = i_M : M \hookrightarrow N$ inclusione

M varietà differenziabile
$\rightsquigarrow TM = \sqcup_{p\in M} T_pM$ <u>spazio tangente</u> a M
$\tau_M : TM \to M$ proiezione definita da $\tau_M(T_pM) = p \ \ \forall\, p \in M$
<u>fibrato tangente</u> di M

<u>Note</u>: 1) $A \subset R^m$ aperto $\Rightarrow TA \cong A \times R^m$ $(v = \overrightarrow{\bar{x}x} \leftrightarrow (\bar{x}, x - \bar{x}))$

2) $(A, \varphi) \in \mathcal{S}_M$ carta locale di $M \rightsquigarrow$
$T\varphi = \sqcup\, T_p\varphi : TA \to T\varphi(A) \cong \varphi(A) \times R^m \subset R^{2m}$

3) $\mathcal{A}$ atlante diff. di $M \rightsquigarrow T\mathcal{A} = \{(TA, T\varphi) \mid (A, \varphi) \in \mathcal{A}\}$
$T\mathcal{A}$ atlante diff. orientato per TM con la top. definita da: $S \subset TM$ aperto $\Leftrightarrow T\varphi(S \cap TA)$ aperto in R^{2m}
((A, φ) con coord. $(x^1, \ldots, x^m) \rightsquigarrow T\varphi : (p, v) \mapsto (x^i, v_x^i)$
(B, ψ) con coord. $(y^1, \ldots, y^m) \rightsquigarrow T\psi : (p, v) \mapsto (y^j, v_y^j)$

$$J(T\psi \circ T\varphi^{-1}) = \left(\begin{array}{c|c} J(\psi \circ \varphi^{-1}) & 0 \\ \hline \partial v_y^j / \partial x^i & J(\psi \circ \varphi^{-1}) \end{array}\right))$$

4) TM $2m$-varietà diff. orientata con $\mathcal{O}_{TM} = \mathcal{O}(T\mathcal{S}_M)$
$\tau_M : TM \to M$ fibrato vettoriale localmente banale
(appl. diff. regolare t.c. $\tau_{M|\tau_M^{-1}(A)} \cong \pi : TA \cong A \times R^m \to A$)

5) $f : M \to N$ appl. diff. (regolare) con M e N varietà diff.
$\rightsquigarrow Tf = \sqcup_{p\in M} T_pf : TM \to TN$ appl. diff. (regolare)
<u>applicazione tangente</u> a f

Teorema di immersione di Whitney

M m-varietà diff. compatta (vale per ogni varietà diff.)

$\exists\, M \hookrightarrow R^{2m+1}$ immersione differenziabile regolare

Dim. $M \subset R^n$ (teorema di immersione differenziabile)

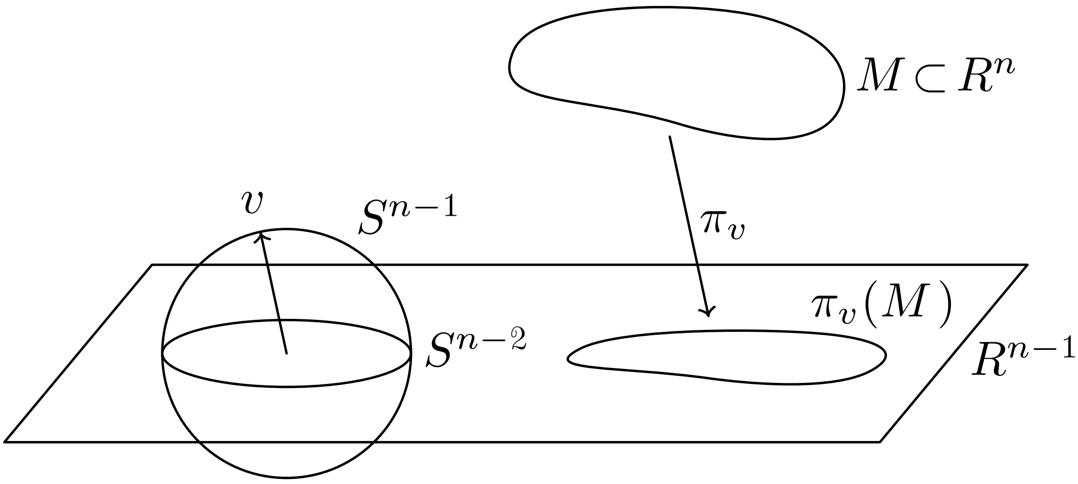
$\rightsquigarrow \Phi : M \times M - \Delta M \to S^{n-1}$ def. $\Phi(p,q) = (p-q)/\|p-q\|$

$\Psi : T_1 M \to S^{n-1}$ def. $\Psi(v) = v$ (applicato nell'origine)

$n > 2m+1 \Rightarrow \operatorname{Im}\Phi$ e $\operatorname{Im}\Psi$ hanno di misura nulla in S^{n-1}

(teorema di Sard con domini di dim $\leq 2m$)

$\Rightarrow \exists\, v \in S^{n-1}$ t.c. $\pi_v : M \to R^{n-1}$ imm. diff. reg.



Campi di vettori

M varietà differenziabile

$V : M \to TM$ campo di vettori su M

$\overset{\text{def}}{\Longleftrightarrow}$ 1) $V(p) = V_p \in T_p M$ per ogni $p \in M$

2) V differenziabile ((A,φ) con coord. $(x^1, \dots, x^m) \rightsquigarrow$

$$V_{|A} = \sum_i v_x^i \frac{\partial}{\partial x^i} \text{ con } v_x^i : A \to R \text{ diff.})$$

Note: 1) $\mathcal{V}M = \{V$ campo vett. su $M\}$ è un $C^\infty(M)$-modulo

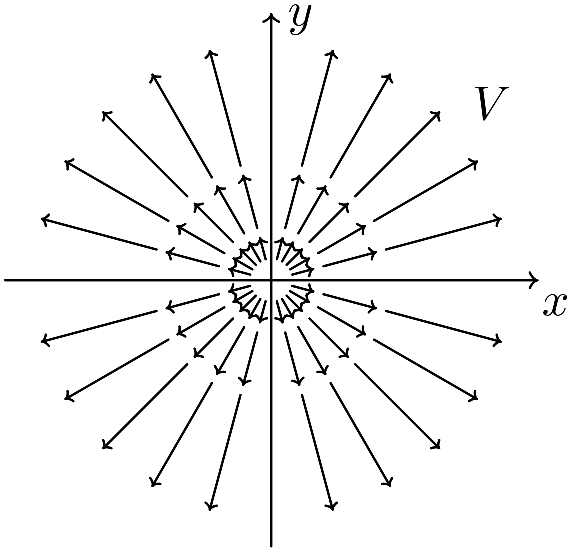
($V, W \in \mathcal{V}M$, $f, g \in C^\infty(M) \rightsquigarrow fV + gW \in \mathcal{V}M$)

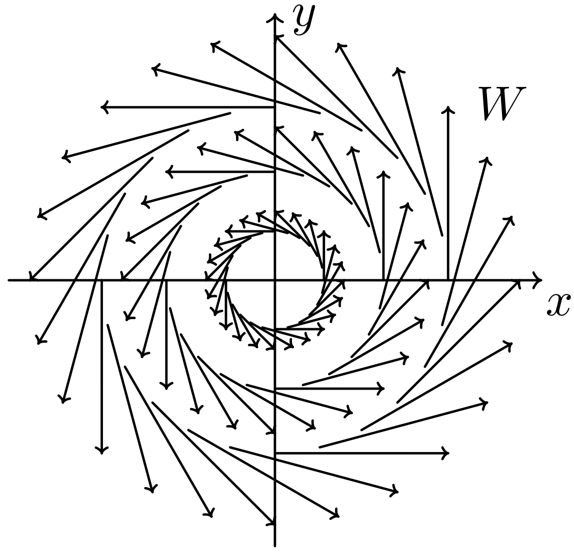
2) V campo vett. su M, $g : M \to R$ funzione diff.

$\rightsquigarrow Vg : M \to R$ funzione diff. definita $Vg(p) = V_p\, g$

$\rightsquigarrow V : C^\infty(M) \to C^\infty(M)$ derivazione

Esempi: $V = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$ e $W = -\,y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$ campi di vett. su R^2





V, W campi di vettori su M, (A, φ) con coord. $(x^1, \dots, x^m)$

$\rightsquigarrow V = \sum_i v^i \dfrac{\partial}{\partial x^i}$ e $W = \sum_j w^j \dfrac{\partial}{\partial x^j}$ nella carta A

$$\rightsquigarrow V(Wg) = \sum_i v^i \frac{\partial}{\partial x^i}\Big(\sum_j w^j \frac{\partial g}{\partial x^j}\Big) = \sum_{i,j} v^i \frac{\partial w^j}{\partial x^i}\frac{\partial g}{\partial x^j} + v^i w^j \frac{\partial^2 g}{\partial x^i \partial x^j}$$

$$W(Vg) = \sum_i w^i \frac{\partial}{\partial x^i}\Big(\sum_j v^j \frac{\partial g}{\partial x^j}\Big) = \sum_{i,j} w^i \frac{\partial v^j}{\partial x^i}\frac{\partial g}{\partial x^j} + w^i v^j \frac{\partial^2 g}{\partial x^i \partial x^j}$$

$$V(Wg) - W(Vg) = \sum_j \sum_i \Big(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i}\Big)\frac{\partial}{\partial x^j} g$$

$\rightsquigarrow [V, W] \overset{\text{def}}{=\!=} VW - WV$ campo di vettori su M

parentesi di Lie di V e W

Note: 1) $[V, W]$ è ben definito su M (non dipende dalle coord.)

2) $[V, W]$ R-bilineare e antisimmetrica rispetto a V e W
$([V, W] = -[W, V] \Rightarrow [V, V] = 0 \;\; \forall\, V, W \in \mathcal{V}M)$

3) $[fV, W] = f[V, W] - (Wf)V \;\; \forall\, f \in C^\infty(M) \; \forall\, V, W \in \mathcal{V}M$

4) $[U, [V, W]] + [V, [W, U]] + [W, [U, V]] = 0 \;\; \forall\, U, V, W \in \mathcal{V}M$

identità di Jacobi (non associatività)

5) $[V, W]_p$ dipende solo da V e W in un intorno di $p \in M$

6) $\Big[\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^j}\Big] = 0 \;\; \forall\, i, j$, ma in generale $[V, W] \neq 0$

$(V, W$ dell'esempio $\rightsquigarrow [V, W] = 0$ ma $\Big[\dfrac{V}{\|V\|}, \dfrac{W}{\|W\|}\Big] \neq 0)$

$f : M \to N$ applicazione diff. con M e N varietà diff.
$V \in \mathcal{V}M$ e $W \in \mathcal{V}N$ f-<u>correlati</u> $\overset{\text{def}}{\iff} T_p f(V_p) = W_{f(p)}\ \forall\, p \in M$

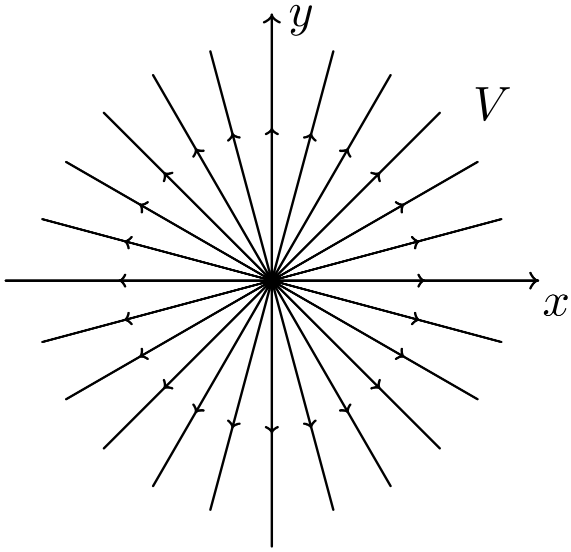
f diffeo $\rightsquigarrow f_* : \mathcal{V}M \to \mathcal{V}N$ unica R-lineare t.c. V e $f_*(V)$ f-correlati
definita $f_*(V)_q = T_p f(V_p)$ con $p = f^{-1}(q)\ \forall\, q \in N$
f diffeo locale $\rightsquigarrow f^* : \mathcal{V}N \to \mathcal{V}M$ unica R-lin. t.c. $f^*(W)$ e W f-corr.
definita $f^*(W)_p = T_p f^{-1}(W_{f(p)})\ \forall\, p \in M$

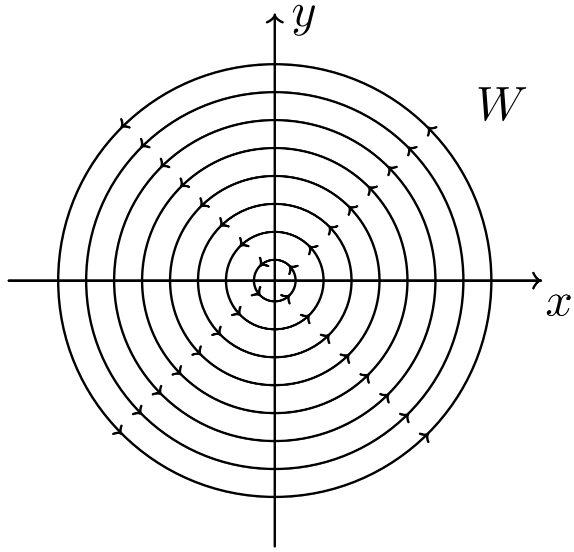
<u>Note</u>: 1) $f_*(gV) = (g \circ f^{-1}) f_*(V)\quad \forall\, g \in C^\infty(M)\ \forall\, V \in \mathcal{V}M$
$f_*([V_1, V_2]) = [f_*(V_1), f_*(V_2)]\quad \forall\, V_1, V_2 \in \mathcal{V}M$
2) $f^*(hW) = (h \circ f) f^*(W)\quad \forall\, h \in C^\infty(N)\ \forall\, W \in \mathcal{V}N$
$f^*([W_1, W_2]) = [f^*(W_1), f^*(W_2)]\quad \forall\, W_1, W_2 \in \mathcal{V}N$

$\gamma : I \to M$ diff. con $I \subset R$ intervallo aperto
<u>curva integrale</u> per V campo di vettori su M
$\overset{\text{def}}{\iff} \gamma'(t) = V(\gamma(t))$ per ogni $t \in I$

<u>Nota</u>: (A, φ) con coord. $(x^1, \ldots, x^m)$
$\rightsquigarrow \gamma(t) = (x^1(t), \ldots, x^m(t))$ curva integ. per $V = \sum_i v^i \dfrac{\partial}{\partial x^i}$
$\Leftrightarrow \dfrac{dx^i}{dt} = v^i(x^1(t), \ldots, x^m(t))\ \forall\, i = 1, \ldots, m$

<u>Esempi</u>: curve int. per i campi vett. V e W dell'esempio sopra

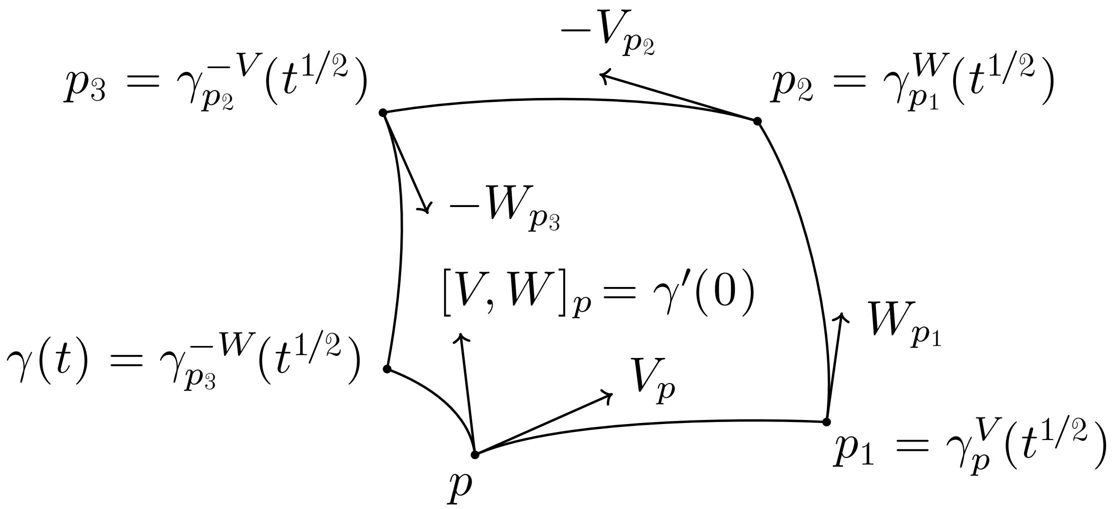




<u>Prop</u>. M varietà differenziabile, V campo di vettori su M
$\forall\, p \in M\ \exists!\, \gamma_p^V : I \to M$ curva integrale massimale per V
uscente da p $(\gamma(0) = p \rightsquigarrow V_p = [\gamma_p^V])$

<u>Dim</u>. $\exists !$ soluzione massimale al problema di Cauchy
$dx^i/dt = v^i(x^1, \ldots, x^m)$, $x^i(0) = \bar{x}^i$ con $i = 1, \ldots, m$
$\Rightarrow \exists !$ curva integrale massimale localmente (in $(A, \varphi) \in \mathcal{A}$)
$\Rightarrow \exists !$ curva massimale globale (raccordo delle soluz. locali)

<u>Nota</u>: *interpretazione geometrica di* $[V, W]$ come $\gamma'(0) = [\gamma]$



$(V = \partial/\partial x^i \,, W = \partial/\partial x^j \Rightarrow \gamma(t) = p \Rightarrow [\partial/\partial x^i, \partial/\partial x^j] = 0)$

<u>Campi di riferimenti</u>

M m-varietà diff., $(F_1, \ldots, F_m)$ campi di vettori su $A \subset M$ aperto
<u>campo di riferimenti</u> su A (o anche <u>riferimento locale</u> per M)
$\overset{\text{def}}{\Longleftrightarrow} T_pM = \langle F_1(p), \ldots, F_m(p) \rangle$ (base di T_pM) $\forall\, p \in A$

<u>Nota</u>: (A, φ) carta locale con coord. $(x^1, \ldots, x^m) \rightsquigarrow F_i = \sum_j f_i^j \dfrac{\partial}{\partial x^j}$
$\rightsquigarrow (F_1, \ldots, F_m)$ campo di riferim. $\Leftrightarrow \det(f_i^j(x)) \neq 0 \;\forall\, x$

<u>campo di riferimenti coordinati</u> su A
$\overset{\text{def}}{\Longleftrightarrow} \forall\, p \in A\ \exists\, (B, \psi)$ carta locale intorno a p con coord. $(x^1, \ldots, x^m)$
tale che $(F_{1|B}, \ldots, F_{m|B}) = \left(\dfrac{\partial}{\partial x^1}, \ldots, \dfrac{\partial}{\partial x^m} \right)$

<u>Nota</u>: ogni varietà differenziabile ha campi coordinati locali,
poche hanno campi di riferim. globali (var. <u>parallelizzabili</u>)
per es.: l'unica superficie compatta parallelizzabile è T^2
(coordinate angolari $(\vartheta^1, \vartheta^2) \rightsquigarrow (\partial/\partial\vartheta^1, \partial/\partial\vartheta^2)$),
ma $\nexists$ campo di vettori ovunque $\neq 0$ su S^2 o $T_{g>1}$

Esempio: V, W dell'esempio sopra

$\rightsquigarrow (V, W)$ campo di riferimenti coord. su $R^2 - \{0\}$

$\rightsquigarrow \left(\frac{V}{\|V\|}, \frac{W}{\|W\|}\right)$ campo di rif. non coord. su $R^2 - \{0\}$

Prop. M varietà diff., $(F_1, \ldots, F_m)$ campo di riferimenti su $A \subset M$ campo di riferimenti coord. $\Leftrightarrow [F_i, F_j] = 0 \;\; \forall\, i, j = 1, \ldots, m$

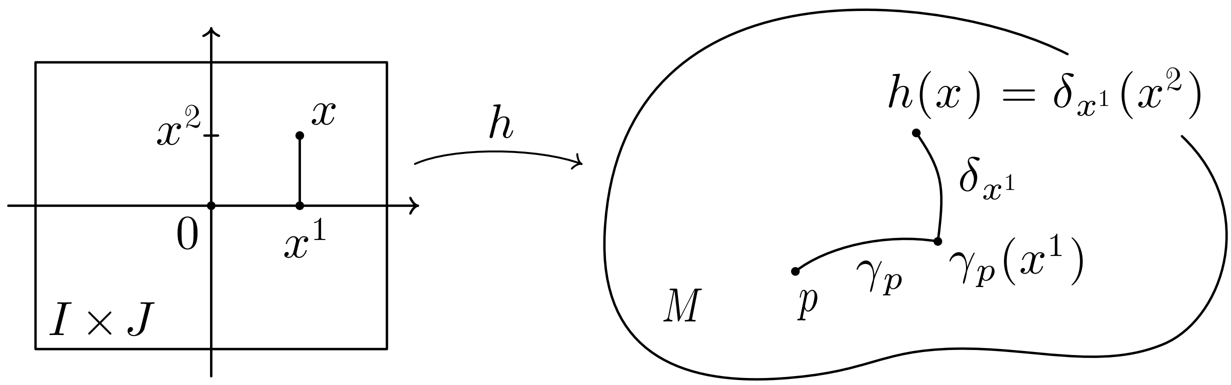
Dim. $\Rightarrow$) $[\partial/\partial x^i, \partial/\partial x^j] = 0 \rightsquigarrow [F_i, F_j] \;\; \forall\, i, j = 1, \ldots, m$

$\Leftarrow$) dimostrazione per $m = 2$ (per $m > 2$ è analoga)

$p \in M \rightsquigarrow \gamma_p : I \to M$ curva int. per F_1 t.c. $\gamma_p(0) = p$

$t \in I \rightsquigarrow \delta_t : J_t \to M$ curva int. per F_2 t.c. $\delta_t(0) = \gamma_p(t)$

$\rightsquigarrow h : I \times J \to M$ appl. diff. definita $h(x^1, x^2) = \delta_{x^1}(x^2)$



$T_x h(\partial/\partial x^2) = F_2(h(x)) \;\forall\, x \in I \times J$

$T_{(x^1,0)} h(\partial/\partial x^1) = F_1(h(x^1, 0)) \;\forall\, x^1 \in I$

$\Rightarrow h$ regolare in $0 \rightsquigarrow h_{|I' \times J'}$ diffeo (teor. funzione inv.)

$\rightsquigarrow \psi = h_|^{-1} : B \to \varphi(B) = I' \times J'$ con coord. $(x^1, \ldots, x^m)$

posto $T_x h(\partial/\partial x^1) = a^1(h(x)) F_1(h(x)) + a^2(h(x)) F_2(h(x))$

si ha: $[a^1 F_1 + a^2 F_2, F_2] = 0$, $[F_1, F_2] = 0 \Rightarrow F_2 a^i = 0$

$\Rightarrow T_x h(\partial/\partial x^1) = F_1(h(x)) \;\forall\, x \in I' \times J'$

Spazi cotangenti

M m-varietà differenziabile, $p \in M$

$\rightsquigarrow T_p^* M \overset{\text{def}}{=\!=} (T_p M)^* = \{l : T_p M \to R \mid l \text{ lineare}\}$

spazio cotangente a M in p

Nota: (A,φ) con coord. $(x^1,\dots,x^m) \rightsquigarrow T_p^*M = \langle dx^1,\dots,dx^m\rangle$

(B,ψ) con coord. $(y^1,\dots,y^m) \rightsquigarrow T_p^*M = \langle dy^1,\dots,dy^m\rangle$

$l = \Sigma_i l_i^x dx^i = \Sigma_j l_j^y dy^j$ con $l_i^x = l\left(\dfrac{\partial}{\partial x^i}\right)$ e $l_j^y = l\left(\dfrac{\partial}{\partial y^j}\right)$

$\rightsquigarrow l_j^y = \sum_i \dfrac{\partial x^i}{\partial y^j}\, l_i^x \quad \left(dy^j = \sum_i \dfrac{\partial y^j}{\partial x^i}\, dx^i\right)$

$f: M \to N$ appl. diff. con M e N varietà diff., $p \in M$, $q = f(p)$

$\rightsquigarrow T_p^* f = (T_p f)^* : T_q^* N \to T_p^* M$ definita $l \mapsto k = T_p^* f(l) \overset{\text{def}}{=\!=} l \circ T_p f$

$\hookrightarrow$ <u>applicazione cotangente</u> a f in p

<u>Note</u>: 1) $T_p^*\mathrm{id}_M = \mathrm{id}_{T_p^*M}$, $T_p^*(g\circ f) = T_p^* f \circ T_{f(p)}^* g$, $\mathrm{rg}_p f = \mathrm{rg}\, T_p^* f$

2) (A,φ) con coord. $(x^1,\dots,x^m) \rightsquigarrow T_p^*M = \langle dx^1,\dots,dx^m\rangle$

(B,ψ) con coord. $(y^1,\dots,y^m) \rightsquigarrow T_q^*N = \langle dy^1,\dots,dy^m\rangle$

$k_i^x = \sum_j \dfrac{\partial y^j}{\partial x^i}\, l_j^y$ $((J_{\varphi(p)}(\psi\circ f\circ\varphi^{-1}))^*$ matrice di $T_p^* f$ risp. alle basi canon. duali indotte da φ e ψ)

M varietà differenziabile

$\rightsquigarrow T^*M = \sqcup_{p\in M} T_p^*M$ <u>spazio cotangente</u> a M

$\tau_M^* : T^*M \to M$ proiezione definita da $\tau_M^*(T_p^*M) = p \ \ \forall p \in M$

$\hookleftarrow$ <u>fibrato cotangente</u> di M

<u>Note</u>: 1) $A \subset R^m$ aperto $\Rightarrow T^*A \cong A \times R^m$

2) $(A,\varphi) \in \mathcal{S}_M$ carta locale di $M \rightsquigarrow$

$(T^*\varphi)^{-1} = \sqcup (T_p^*\varphi)^{-1} : T^*A \to T^*\varphi(A) \cong \varphi(A)\times R^m \subset R^{2m}$

3) $\mathcal{A}$ atlante diff. di $M \rightsquigarrow T^*\mathcal{A} = \{(T^*A, (T^*\varphi)^{-1}) \mid (A,\varphi)\in\mathcal{A}\}$

$T^*\mathcal{A}$ atlante diff. orientato per T^*M con la top. definita da: $S \subset T^*M$ aperto $\Leftrightarrow (T^*\varphi)^{-1}(S\cap T^*A)$ aperto in R^{2m}

((A,φ) con coord. $(x^1,\dots,x^m) \rightsquigarrow (T^*\varphi)^{-1}: (p,l) \mapsto (x^i, l_i^x)$

(B,ψ) con coord. $(y^1,\dots,y^m) \rightsquigarrow (T^*\psi)^{-1}: (p,l) \mapsto (y^j, l_j^y)$

$$J((T^*\psi)^{-1}\circ T^*\varphi) = \left(\begin{array}{c|c} J(\psi\circ\varphi^{-1}) & 0 \\ \hline \partial l_y^j/\partial x^i & J(\varphi\circ\psi^{-1})^* \end{array}\right))$$

4) T^*M $2m$-varietà diff. orientata con $\mathcal{O}_{T^*M} = \mathcal{O}(T^*\mathcal{S}_M)$

$\tau_M^* : T^*M \to M$ fibrato vettoriale localmente banale

5) $f: M \to N$ appl. diff. $\not\rightsquigarrow T^*f : T^*N \to T^*M$ appl. diff.

$g : M \to R$ funzione diff., $p \in M$
$\rightsquigarrow d_p g : T_pM \to R$ definita $v \mapsto d_p g(v) \stackrel{\text{def}}{=} v\, g$
differenziale di g in p

Note: 1) $d_p g \in T_p^* M$ (dg è lineare)
2) in coord. locali $(x^1, \ldots, x^m)$ si ha: $d_p g = \sum_i \dfrac{\partial g}{\partial x^i}\, dx^i$

Forme differenziali

M varietà differenziabile
$L : M \to T^*M$ forma differenziale lineare su M
$\stackrel{\text{def}}{\iff}$ 1) $L(p) = L_p \in T_p^*M$ per ogni $p \in M$
2) L differenziabile ((A, φ) con coord. $(x^1, \ldots, x^m) \rightsquigarrow$
$L_{|A} = \sum_i l_i^x dx^i$ con $l_i^x : A \to R$ diff.)

Note: 1) $\mathcal{L}M = \{L \text{ f.d.l. su } M\}$ è un $C^\infty(M)$-modulo
($L, K \in \mathcal{L}M$, $f, g \in C^\infty(M) \rightsquigarrow fL + gK \in \mathcal{L}M$)
2) L forma diff. lin. su M, V campo di vettori su M
$\rightsquigarrow LV : M \to R$ funzione diff. definita $LV(p) = L_p(V_p)$
3) $g : M \to R$ funz. diff. $\rightsquigarrow dg$ f.d.l. (differenziale di g)
definita $dg(p) = d_p g : T_pM \to R$

$f : M \to N$ applicazione diff. con M e N varietà diff.
$\rightsquigarrow f^* : \mathcal{L}N \to \mathcal{L}M$ applicazione R-lineare
definita $(f^*L)_p = T_p^* f(L_{f(p)})\ \forall\, p \in M$

Note: 1) $f^*(gL) = (g \circ f) f^*(L)\ \ \forall\, g \in C^\infty(N)\ \forall\, L \in \mathcal{L}N$
2) f diffeo loc. $\Rightarrow (f^*L)(f^*V) = LV \circ f\ \ \forall\, L \in \mathcal{L}N\ \forall\, V \in \mathcal{V}N$

M varietà diff., $p \in M$
$\Lambda_p^k M \stackrel{\text{def}}{=} \{\lambda : (T_pM)^k \to R \mid \lambda \text{ multilineare antisimmetrica}\}$
$\Lambda_p M \stackrel{\text{def}}{=} \oplus_{k \geq 0} \Lambda_p^k M$ algebra di Grassmann di M in p

$\lambda \in \Lambda_p^k M$, $\mu \in \Lambda_p^h M \rightsquigarrow \lambda \wedge \mu \in \Lambda_p^{k+h} M$ prodotto esterno

$$\lambda \wedge \mu(v_1, \ldots, v_{k+h}) \stackrel{\text{def}}{=} \sum_{\substack{\pi = (i_1, \ldots, i_{k+h}) \\ i_1 < \ldots < i_k\,,\, i_{k+1} < \ldots < i_{k+h}}} \sigma(\pi)\, \lambda(v_{i_1}, \ldots, v_{i_k})\, \mu(v_{i_{k+1}}, \ldots, v_{i_{k+h}})$$

Note: 1) $\Lambda^0_p M \cong R$, $\Lambda^1_p M \cong T^*_p M$, $\Lambda^k_p M$ spazio vett. reale $\forall\, k \geq 0$

$\Lambda_p M$ algebra graduata anticomm. $(\lambda \wedge \mu = (-1)^{kh} \mu \wedge \lambda)$

2) (A, φ) con coord. $(x^1, \ldots, x^m)$,

$I = i_1 < \ldots < i_k$, $J = j_1 < \ldots < j_k$

$\rightsquigarrow dx^I \stackrel{\text{def}}{=\!=} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, $\dfrac{\partial}{\partial x^J} = \Big(\dfrac{\partial}{\partial x^{j_1}}, \ldots, \dfrac{\partial}{\partial x^{j_k}}\Big)$

$\Rightarrow dx^I\Big(\dfrac{\partial}{\partial x^J}\Big) = \delta^I_J$ (vale 1 se $I = J$, 0 altrimenti)

3) (A, φ) con coord. $(x^1, \ldots, x^m) \rightsquigarrow \Lambda^k_p M = \big\langle \{dx^I \mid |I| = k\} \big\rangle$

(B, ψ) con coord. $(y^1, \ldots, y^m) \rightsquigarrow \Lambda^k_p M = \big\langle \{dy^J \mid |J| = k\} \big\rangle$

$\lambda = \sum\limits_{|I|=k} \lambda^x_I dx^I = \sum\limits_{|J|=k} \lambda^y_J dy^J$ con $\lambda^x_I = \lambda\Big(\dfrac{\partial}{\partial x^I}\Big)$, $\lambda^y_J = \lambda\Big(\dfrac{\partial}{\partial y^J}\Big)$

$\rightsquigarrow \lambda^y_J = \sum\limits_{|I|=k} \det\Big(\dfrac{\partial x^I}{\partial y^J}\Big) \lambda^x_I \quad \Big(dy^J = \sum\limits_{|I|=k} \det\Big(\dfrac{\partial y^J}{\partial x^I}\Big) dx^I\Big)$

4) $\dim \Lambda^k_p M = \dbinom{m}{k} \;\; \forall\, k \geq 0 \rightsquigarrow \dim \Lambda_p M = 2^m$

$\Lambda^m_p M = \big\langle dx^1 \wedge \ldots \wedge dx^m \big\rangle \cong R$, $\Lambda^k_p M \cong 0 \;\; \forall\, k > m$

$(dx^1 \wedge \ldots \wedge dx^m (v_1, \ldots, v_m) = \det(v^j_i) =$ vol. eucl. orient.)

M varietà diff. $\rightsquigarrow \Lambda^k M \stackrel{\text{def}}{=\!=} \cup_{p \in M} \Lambda^k_p M$, $\Lambda M \stackrel{\text{def}}{=\!=} \cup_{p \in M} \Lambda_p M$

$\lambda : M \to \Lambda^k M$ <u>k-forma differenziale</u> su M

$\stackrel{\text{def}}{\Longleftrightarrow}$ 1) $\lambda(p) = \lambda_p \in \Lambda^k_p M$ per ogni $p \in M$

2) λ "differenziabile" ((A, φ) con coord. $(x^1, \ldots, x^m) \rightsquigarrow$

$\lambda_{|A} = \sum\limits_{|I|=k} \lambda^x_I dx^I$ con $\lambda^x_I : A \to R$ diff.)

$\mathcal{D}^k M \stackrel{\text{def}}{=\!=} \{k\text{-forme differenziabili su } M\}$

$\mathcal{D} M \stackrel{\text{def}}{=\!=} \oplus_{k \geq 0} \mathcal{D}^k M$ <u>algebra di Grassmann</u> di M

Note: 1) $\mathcal{D}^0 M \cong C^\infty(M)$, $\mathcal{D}^1 M \cong \mathcal{L} M$

$\mathcal{D}^k M$ è un $C^\infty(M)$-modulo $\forall\, k \geq 0$

$(\lambda, \mu \in \mathcal{D}^k M$, $f, g \in C^\infty(M) \rightsquigarrow f\lambda + g\mu \in \mathcal{D}^k M)$

2) $\mathcal{D} M$ algebra graduata anticomm. $((\lambda \wedge \mu)_p \stackrel{\text{def}}{=\!=} \lambda_p \wedge \mu_p)$

$f: M \to N$ applicazione diff. con M e N varietà diff.
$\rightsquigarrow f^*: \mathcal{D}^k N \to \mathcal{D}^k M$ per ogni $k \geq 0$, definita dall'uguaglianza
$f^*(\lambda)_p(v_1, \ldots, v_k) = \lambda_{f(p)}(T_p f(v_1), \ldots, T_p f(v_k)) \ \forall\, p \in M, v_i \in T_p M$

Note: 1) $f^*: \mathcal{D}^0 N \to \mathcal{D}^0 M$ coincide con $g \mapsto g \circ f \ \ \forall\, g \in C^\infty(N)$
$f^*: \mathcal{D}^1 N \to \mathcal{D}^1 M$ coincide con $f^*: \mathcal{L}N \to \mathcal{L}M$ def. sopra
2) f^* è R-lineare e $f^*(\lambda\wedge\mu) = f^*(\lambda)\wedge f^*(\mu) \ \forall\, \lambda, \mu \in \mathcal{D}^{k,h} N$
3) in coordinate locali $(x^1, \ldots, x^m)$ di M e $(y^1, \ldots, y^n)$ di N

$$\lambda = \sum_J \lambda^y_J dy^J \mapsto f^*(\lambda) = \sum_I \lambda^x_I dx^I \text{ con } \lambda^x_I = \sum_J \det\Big(\frac{\partial y^J}{\partial x^I}\Big)\lambda^y_J$$

4) $(\mathrm{id}_M)^* = \mathrm{id}_{\mathcal{D}^k M}$, $(g \circ f)^* = f^* \circ g^*$
5) $i: N \subset M$ sottovar. diff., $\lambda \in \mathcal{D}^k M \rightsquigarrow \lambda_{|N} = i^*(\lambda) \in \mathcal{D}^k N$

Prop. M varietà diff., $\exists!\, d: \mathcal{D}M \to \mathcal{D}M$ (differenziale esterno)
t.c.: 1) d appl. R-lineare e $d(\mathcal{D}^k M) \subset \mathcal{D}^{k+1} M \ \forall\, k \geq 0$
2) $d_{|\mathcal{D}^0 M}: \mathcal{D}^0 M \to \mathcal{D}^1 M$ = differenziale di funzioni
3) $d(\lambda\wedge\mu) = d\lambda\wedge\mu + (-1)^k \lambda\wedge d\mu \ \ \forall\, \lambda \in \mathcal{D}^k M$, $\mu \in \mathcal{D}^h M$
4) $d \circ d = 0$

Dim. Caso speciale $M = A \subset R^m$
$d: \lambda = \sum_I \lambda_I dx^I \mapsto \sum_I d\lambda_I \wedge dx^I$ con $d\lambda_I$ diff. di funzione
soddisfa le proprietà 1 e 2 banalmente e anche 3 e 4:

$$\begin{aligned} 3)\ d(\lambda\wedge\mu) &= d(\textstyle\sum_{I,J} \lambda_I \mu_J dx^I \wedge dx^J) = \sum_{I,J} d(\lambda_I \mu_J)\wedge dx^I \wedge dx^J \\ &= \textstyle\sum_{I,J} d\lambda_I \wedge dx^I \wedge \mu_J dx^J + (-1)^{|I|} \lambda_I dx^I \wedge d\mu_J \wedge dx^J \\ &= d\lambda\wedge\mu + (-1)^k \lambda d\mu \end{aligned}$$

$$4)\ dd\lambda = dd(\textstyle\sum_I \lambda_I dx^I) = d(\sum_I d\lambda_I \wedge dx^I) = d(\sum_{I,i} \dfrac{\partial \lambda_I}{\partial x^i} dx^i \wedge dx^I)$$

$$= \textstyle\sum_{I,i} d\, \dfrac{\partial \lambda_I}{\partial x^i} \wedge dx^i \wedge dx^I = \sum_{I,i,j} \dfrac{\partial^2 \lambda_I}{\partial x^i \partial x^j} dx^j \wedge dx^i \wedge dx^I = 0$$

$\bar{d}: \mathcal{D}A \to \mathcal{D}A$ altra appl. con le stesse proprietà
$\Rightarrow \bar{d}dx^i = \bar{d}\bar{d}x^i = 0 \ \ \forall\, i = 1, \ldots, m$
$\Rightarrow \bar{d}dx^I = \bar{d}(dx^{i_1}\wedge \ldots \wedge dx^{i_k}) = 0 \ \ \forall\, I = i_1 < \ldots < i_k$
$\Rightarrow \bar{d}\lambda = \bar{d}(\sum_I \lambda_I dx^I) = \sum_I (\bar{d}\lambda_I \wedge dx^I + \lambda_I \bar{d}dx^I) = \sum_I d\lambda_I \wedge dx^I$

<u>Caso generale/unicità</u> (se d esiste allora è unico)

$d : \mathcal{D}M \to \mathcal{D}M$ soddisfa le proprietà $1, \ldots, 4$

$\Rightarrow d$ è locale, cioè: $\lambda_{|A} = \mu_{|A} \Rightarrow (d\lambda)_{|A} = (d\mu)_{|A}$
$\forall\, \lambda, \mu \in \mathcal{D}M\ \forall\, A \subset M$ aperto

$(p \in A \rightsquigarrow A'$ intorno aperto di p t.c. $A' \subset \mathrm{Cl}\, A' \subset A$
$f \in C^\infty(M)$ t.c. $f(A') = 1$ e $f(M - A) = 0$

$\lambda_{|A} = \mu_{|A} \Rightarrow f\lambda = f\mu \Rightarrow d(f\lambda) = d(f\mu)$

$\Rightarrow (df \wedge \lambda + f d\lambda)_p = (df \wedge \mu + f d\mu)_p \Rightarrow (d\lambda)_p = (d\mu)_p)$

$\Rightarrow d_A : \mathcal{D}A \to \mathcal{D}A$ definita $(d_A\lambda)_p = (d(f\lambda))_p\ \forall\, p \in A$
è ben-definita (non dipende dalla scelta di f)
e soddisfa le proprietà $1, \ldots, 4$ (sono tutte locali)

$\Rightarrow d_A$ unica se $A \cong \varphi(A) \subset R^m$ (carta locale)

$\Rightarrow d$ unica $((d\lambda)_p = (d(f\lambda))_p = d_A(\lambda_{|A})_p\ \forall\, p \in (A, \varphi))$

<u>Caso generale/esistenza</u>

(A, φ) con coord. $(x^1, \ldots, x^m) \rightsquigarrow d^x : \mathcal{D}A \to \mathcal{D}A$

(B, ψ) con coord. $(y^1, \ldots, y^m) \rightsquigarrow d^y : \mathcal{D}B \to \mathcal{D}B$

$\Rightarrow (d^x(\lambda_{|A}))_{|A\cap B} = d^x(\lambda_{|A\cap B}) = d^y(\lambda_{|A\cap B}) = (d^y(\lambda_{|B}))_{|A\cap B}$

$\rightsquigarrow d : \mathcal{D}M \to \mathcal{D}M$ definita $(d\lambda)_{|A} = d_A(\lambda_{|A})\ \forall\, (A, \varphi) \in \mathcal{S}_M$
è ben definita e soddisfa le proprietà $1, \ldots, 4$

<u>Prop</u>. M varietà differenziabile, $\lambda \in \mathcal{L}M$, $V, W \in \mathcal{V}M$
$\Rightarrow d\lambda(V, W) = V\lambda(W) - W\lambda(V) - \lambda([V, W])$

<u>Dim</u>. (A, φ) carta locale con coord. $(x^1, \ldots, x^m)$

$\rightsquigarrow \lambda = \sum_i \lambda_i dx^i$, $V = \sum_j v^j \dfrac{\partial}{\partial x^j}$, $W = \sum_k w^k \dfrac{\partial}{\partial x^k}$

$\rightsquigarrow d\lambda = \sum_{i,j} \dfrac{\partial \lambda_i}{\partial x^j} dx^j \wedge dx^i$, $d\lambda(V, W) = \sum_{i,j} \dfrac{\partial \lambda_i}{\partial x^j}(v^j w^i - v^i w^j)$

$V\lambda(W) = V \sum_i (\lambda_i w^i) = \sum_{i,j} v^j \dfrac{\partial \lambda_i}{\partial x^j} w^i + v^j \lambda_i \dfrac{\partial w^i}{\partial x^j}$

$W\lambda(V) = W \sum_i (\lambda_i v^i) = \sum_{i,j} w^j \dfrac{\partial \lambda_i}{\partial x^j} v^i + w^j \lambda_i \dfrac{\partial v^i}{\partial x^j}$

$\lambda([V, W]) = \sum_{i,j} \lambda_i \Big(v^j \dfrac{\partial w^i}{\partial x^j} - w^j \dfrac{\partial v^i}{\partial x^j} \Big)$

$\lambda \in \mathcal{D}^k M$ k-forma diff. <u>chiusa</u> $\overset{\text{def}}{\Longleftrightarrow} d\lambda = 0$ (cioè $\lambda \in \operatorname{Ker} d$)
k-forma diff. <u>esatta</u> $\overset{\text{def}}{\Longleftrightarrow} \lambda = d\mu$ (cioè $\lambda \in \operatorname{Im} d$)

<u>Note</u>: 1) k-forma diff. esatta $\not\Leftarrow\Rightarrow$ k-forma diff. chiusa
2) $f \in \mathcal{D}^0 M \cong C^\infty(M)$ chiusa $\Leftrightarrow$ f localmente costante
3) $\lambda \in \mathcal{D}^1 M \rightsquigarrow \lambda_{|A} = \sum_i \lambda_i dx^i$ con $(A, \varphi) \in \mathcal{S}_M$

$$\rightsquigarrow d\lambda_{|A} = \sum_{i<j} \Big(\frac{\partial \lambda_j}{\partial x^i} - \frac{\partial \lambda_i}{\partial x^j}\Big) dx^i \wedge dx^j$$

λ chiusa $\Leftrightarrow \dfrac{\partial \lambda_j}{\partial x^i} = \dfrac{\partial \lambda_i}{\partial x^j}$ $\forall\, i, j = 1, \ldots, m$ $\forall\, (A, \varphi) \in \mathcal{A}$

λ esatta $\Leftrightarrow \exists\, f \in C^\infty(M)$ t.c. $\lambda_i = \dfrac{\partial f}{\partial x^i}$ $\forall\, i$ $\forall\, (A, \varphi) \in \mathcal{A}$

<u>Esempi</u>: 1) $d\vartheta \in \mathcal{D}^1 S^1$ è chiusa ma non esatta
(ϑ è ben definita localmente ma non su tutto S^1)
2) $-y/(x^2+y^2)dx + x/(x^2+y^2)dy \in \mathcal{D}^1(R^2 - \{0\})$
è chiusa ma non esatta (non si può estendere a R^2)

<u>Note</u>: 1) $f : M \to N$ appl. diff. $\Rightarrow f^*(d\lambda) = d(f^*(\lambda))$ $\forall\, \lambda \in \mathcal{D}^k N$
2) $\lambda \in \mathcal{D}^k N$ chiusa/esatta $\Rightarrow f^*(\lambda)$ chiusa/esatta

<u>Lemma di Poincaré</u> $\lambda \in \mathcal{D}^k R^m$ chiusa $\Rightarrow$ esatta $\forall\, 1 \le k \le m$

<u>Dim</u>. $m = 1 \Rightarrow k = 1 \Rightarrow \lambda \in \mathcal{D}^1 R$ chiusa ed esatta
($\lambda = df$ con $f \in C^\infty(R)$ definita $f(x) = \int_0^x \lambda$)
$m > 1 \rightsquigarrow \Phi^k : \mathcal{D}^k R^m \to \mathcal{D}^{k-1} R^m$ definita per ogni $k \ge 1$
$\lambda = \sum_I \lambda_I dx^I \mapsto \Phi^k(\lambda) = (-1)^{k-1} \sum_{m \in I} \widehat{\lambda}_I dx^{\widehat{I}}$
dove: $\widehat{f}(x) = \int_0^{x^m} f(x^1, \ldots, x^{m-1}, \xi) d\xi$, $\widehat{I} = I - \{m\}$

$$\Rightarrow d(\Phi^k(\lambda)) = (-1)^{k-1} \sum_{m \in I} \sum_{j \notin I} \frac{\partial \widehat{\lambda}_I}{\partial x^j} dx^j \wedge dx^{\widehat{I}} + \sum_{m \in I} \lambda_I dx^I$$

$$\Phi^{k+1}(d\lambda) = (-1)^k \sum_{m \in I} \sum_{j \notin I} \widehat{\frac{\partial \lambda_I}{\partial x^j}} dx^j \wedge dx^{\widehat{I}} + \sum_{m \notin I} \lambda_I dx^I - \mu$$

$$\text{con } \mu = \sum_{m \notin I} \lambda_I(x^1, \ldots, x^{m-1}, 0) dx^I = \pi^*(\lambda_{|R^{m-1}})$$

$$\Rightarrow d(\Phi^k(\lambda)) + \Phi^{k+1}(d\lambda) = \lambda - \mu$$

$$\begin{aligned} d\lambda = 0 &\Rightarrow d\lambda_{|R^{m-1}} = 0 \Rightarrow \mu = d\nu \text{ (induzione)} \\ &\Rightarrow \lambda = d(\Phi^k(\lambda)) + d\nu = d(\Phi^k(\lambda) + \nu) \end{aligned}$$

<u>Corol</u>. M varietà diff., $k \geq 1$, $\lambda \in \mathcal{D}^k M$ chiusa $\Leftrightarrow$ localm. esatta
($\forall\, p \in M\ \exists\, A \subset M$ intorno aperto di p tale che $\lambda_{|A}$ è esatta)

<u>Dim</u>. $p \in M \rightsquigarrow (A, \varphi)$ carta locale speciale intorno a p
λ chiusa $\Rightarrow \lambda_{|A}$ chiusa $\Rightarrow \lambda_{|A}$ esatta ($A \cong R^m$)

<u>Integrazione</u>

$C \subset M$ <u>chiuso ammissibile</u> in M m-varietà diff.
$\overset{\text{def}}{\Longleftrightarrow}$ 1) C chiuso regolare (cioè $C = \mathrm{Cl}(\mathrm{Int}\, C)$)
2) $\mathrm{Fr}\, C = F_1 \cup \ldots \cup F_l$ con $F_i \subset C \subset M$ chiuso
3) F_i ch. ammissibile in $N_i \subset M$ $(m{-}1)$-sottovar. diff.
4) $\mathrm{Int}_{N_i} F_i \cap \mathrm{Int}_{N_j} F_j = \varnothing$ per ogni $i \neq j$

<u>Note</u>: 1) per $m = 0$ ogni $C \subset M$ è chiuso ammissibile
2) M chiuso ammissibile in se stesso con $\mathrm{Fr}\, M = \varnothing$
3) $C \subset M$ con $\mathrm{Fr}\, C = N \subset M$ $(m-1)$-sottovar. diff.
<u>varietà con bordo</u> (localmente $\cong R^m, R^m_+$)

$C \subset A \subset R^m$ chiuso ammissibile (in A aperto in R^m)
$f : A \to R$ continua con $\mathrm{supp}(f) \subset B = I_1 \times \ldots \times I_m$
$\rightsquigarrow \widetilde{f} : B \to R$ estensione di $f_{|C}$ definita $\widetilde{f}(x) = 0\ \forall\, x \in B - C$
$\rightsquigarrow \int_C f(x^1, \ldots, x^m) dx^1 \ldots dx^m = \int_{I_m} \ldots \int_{I_1} \widetilde{f}(x^1, \ldots, x^m) dx^1 \ldots dx^m$
<u>integrale di Riemann</u> di f su C

<u>Note</u>: 1) $\int_C f\, dx^1 \ldots dx^m$ è ben definito (non dipende da B)
e si può cambiare l'ordine d'integrazione (t. di Fubini)
2) $\int_C (af{+}bg) dx^1 \ldots dx^m = a\int_C f\, dx^1 \ldots dx^m + b\int_C g\, dx^1 \ldots dx^m$
3) $f \leq g \Rightarrow \int_C f\, dx^1 \ldots dx^m \leq \int_C g\, dx^1 \ldots dx^m$
4) $C_1, C_2 \subset A$ t.c. $C = C_1 \cup C_2$ e $\mathrm{Int}\, C_1 \cap \mathrm{Int}\, C_2 = \varnothing$
$\Rightarrow \int_C f\, dx^1 \ldots dx^m = \int_{C_1} f\, dx^1 \ldots dx^m + \int_{C_2} f\, dx^1 \ldots dx^m$
5) $h : A \to A'$ diffeo, $C \subset A$, $C' = h(C)$, $y = h(x)$
$\Rightarrow \int_{C'} f\, dy^1 \ldots dy^m = \int_C f\, |\det(\partial y^j / \partial x^i)| dx^1 \ldots dx^m$

$M = (M, \mathcal{O})$ m-varietà differenziabile <u>orientata</u>,
$C \subset M$ chiuso ammissibile, $\omega \in \mathcal{D}^m M$ con supporto compatto
$\{(A_n, \varphi_n)\} \subset \mathcal{O}$ atlante speciale, $\{f_n\}$ partizione dell'unità subord.
$\rightsquigarrow \int_C \omega \overset{\text{def}}{=\!=} \sum_n \int_{\varphi_n(C \cap A_n)} f_n \omega^x dx^1 \dots dx^m$ <u>integrale</u> di ω su C

<u>Note</u>: 1) $\int_C \omega$ è ben definito (non dipende da $\{(A_n, \varphi_n)\}$ e $\{f_n\}$)
($\{(B_n, \psi_n)\} \subset \mathcal{O}$ atl. spec., $\{g_n\}$ partiz. dell'unità subord.

$$\begin{aligned}
\Rightarrow & \textstyle\sum_{n'} \int_{\psi_{n'}(C \cap B_{n'})} g_{n'} \omega^y dy^1 \dots dy^m \\
&= \textstyle\sum_{n,n'} \int_{\psi_{n'}(C \cap A_n \cap B_{n'})} f_n g_{n'} \omega^y dy^1 \dots dy^m \\
&= \textstyle\sum_{n,n'} \int_{\varphi_n(C \cap A_n \cap B_{n'})} f_n g_{n'} \omega^y \det(\partial y^j / \partial x^i)\, dx^1 \dots dx^m \\
&= \textstyle\sum_{n,n'} \int_{\varphi_n(C \cap A_n \cap B_{n'})} f_n g_{n'} \omega^x dx^1 \dots dx^m \\
&= \textstyle\sum_n \int_{\varphi_n(C \cap A_n)} f_n \omega^x dx^1 \dots dx^m)
\end{aligned}$$

2) $\int_C a\,\omega + b\,\omega' = a \int_C \omega + b \int_C \omega'$ $\forall\, a, b \in R$ $\forall\, \omega, \omega' \in \mathcal{D}^m M$
3) $C_1, C_2 \subset M$ t.c. $C = C_1 \cup C_2$ e $\operatorname{Int} C_1 \cap \operatorname{Int} C_2 = \varnothing$
$\Rightarrow \int_C \omega = \int_{C_1} \omega + \int_{C_2} \omega \quad \forall\, \omega \in \mathcal{D}^m M$
4) orientazione opposta $-\mathcal{O}$ su $M \rightsquigarrow$ integrali opposti

$\omega \in \mathcal{D}^m M$ <u>forma di volume</u> su M m-varietà diff.
$\overset{\text{def}}{\iff} \omega_p \neq 0$ per ogni $p \in M$

<u>Prop</u>. M varietà diff., $\exists$ forma di volume su $M \Leftrightarrow M$ orientabile

<u>Dim</u>. $\Rightarrow$) $\omega \in \mathcal{D}^m M$ forma di volume su $M = (M, \mathcal{S})$
$\rightsquigarrow \{(A, \varphi) \in \mathcal{S} \mid \omega^x > 0$ con $x = (x^1, \dots, x^m)$ coord. su $A\}$
atlante orient. su M $(\omega^x, \omega^y > 0 \Rightarrow \det(\partial y^j / \partial x^i) > 0)$
$\Leftarrow$) $\{(A_n, \varphi_n)\}$ atl. orient. su M, $\{f_n\}$ part. dell'unità subord.
$\rightsquigarrow \omega = \sum_n f_n\, dx^1 \wedge \dots \wedge dx^m$ forma di volume su M

<u>Note</u>: 1) $\{$orientazioni su $M\} \leftrightarrow \{$forme di vol. su $M\}/_{\omega \sim f\omega\ \forall f > 0}$
2) ω forma di vol. su M ($\rightsquigarrow$ orient. su M) $\rightsquigarrow$ misura su M
t.c. $\operatorname{mis}(C) = \int_C \omega$ per ogni $C \subset M$ compatto ammissibile
3) ω forma di vol. su $M \rightsquigarrow C^\infty(M) \leftrightarrow \mathcal{D}^m M$ $(f \leftrightarrow f\,\omega)$

M varietà diff., ω forma di volume su M ($\rightsquigarrow$ orient. su M)
$\rightsquigarrow \int_C f\omega \;\; \forall f \in C^\infty(M)$ con supporto compatto
$\quad$ integrale di f su $C \subset (M,\omega)$

Nota: l'integrazione di funzioni in $M = (M,\omega)$ generalizza l'integrazione di Riemann in $R^m = (R^m, dx^1 \wedge \ldots \wedge dx^m)$ e soddisfa proprietà analoghe a 2 , 3 e 4 viste per questa

M varietà diff., C chiuso amm. in $N \subset M$ n-sottovar. diff. orient.
$\rightsquigarrow \int_C \lambda \overset{\text{def}}{=\!=} \int_C \lambda_{|N} = \int_C i^*\lambda \;\; \forall \lambda \in \mathcal{D}^n M$ con supporto compatto

$\partial C \overset{\text{def}}{=\!=}$ $\mathrm{Fr}_N C = F_1 \cup \ldots \cup F_l$ con $F_i \subset N_i$ $(n-1)$-sottovar. diff. di N orientata in modo che $\forall\,(A,\varphi)$ carta speciale di N tale che $\varphi(A \cap C) = R^n_+ = \{x \in R^n \mid x^n \geq 0\}$ si ha $(A \cap N_i, \varphi_{|A\cap N_i}) \in \mathcal{O}_{N_i} \Leftrightarrow (A,\varphi) \in (-1)^n \mathcal{O}_N$

$\rightsquigarrow \int_{\partial C} \mu \overset{\text{def}}{=\!=} \sum_i \int_{F_i} \mu \;\; \forall \mu \in \mathcal{D}^{n-1}M$ con supporto compatto

Note: 1) $\int_{\partial C} \mu$ è ben definito (non dipende da $F_1, \ldots, F_l$)
2) orient. opposta su $N \rightsquigarrow$ orient. opposta su ogni N_i $\rightsquigarrow$ integrali $\int_C \lambda$ e $\int_{\partial C} \mu$ opposti

Teorema di Stokes

M varietà diff., C ch. amm. in $N \subset M$ n-sottovar. diff. orient.
$\Rightarrow \int_C d\lambda = \int_{\partial C} \lambda \;\; \forall \lambda \in \mathcal{D}^{n-1}M$ con supporto compatto

Dim. si può assumere $N = M$ (sostituendo λ in M con $i^*\lambda$ in N) e $C \subset M$ varietà con bordo (sostituendo M con $M - \cup_i \mathrm{Fr}_{N_i} F_i$)
T. fond. calcolo integrale $\Rightarrow$ casi spec. $M = R^m$ e $C = R^m, R^m_+$
$\{(A_n, \varphi_n)\}$ atlante spec. orient., $\{f_n\}$ part. dell'unità subord.
$\rightsquigarrow \int_C d\lambda = \sum_n \int_{\varphi_n(C\cap A_n)} f_n d\lambda = \sum_n \int_{\varphi_n(C\cap A_n)} d(f_n\lambda)$
$\qquad = \sum_n \int_{\varphi_n(\partial C\cap A_n)} f_n\lambda = \int_{\partial C} \lambda$

Esempio: $C \subset R^2$ chiuso amm., $f : R^2 \to R$ diff. con supp. comp.

$$\Rightarrow \int_{\partial C} f\,dx = -\int_C \frac{\partial f}{\partial y} dx dy \quad \text{e} \quad \int_{\partial C} f\,dy = \int_C \frac{\partial f}{\partial x} dx dy$$

Calcolo vettoriale in R^2

$$
\begin{tikzcd}
C^\infty(R^2) \arrow[r, "\mathrm{grad}"] \arrow[d, equal] & \mathcal{V}R^2 \arrow[r, "\mathrm{rot}"] \arrow[d, leftrightarrow, "\cong"', "1"] & C^\infty(R^2) \arrow[d, leftrightarrow, "\cong"', "2"] \\
\mathcal{D}^0R^2 \arrow[r, "d"'] & \mathcal{D}^1R^2 \arrow[r, "d"'] & \mathcal{D}^2R^2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}R^2 \arrow[r, "\mathrm{div}"] \arrow[d, leftrightarrow, "\cong"', "1'"] & C^\infty(R^2) \arrow[d, leftrightarrow, "\cong"', "2"] \\
\mathcal{D}^1R^2 \arrow[r, "d"'] & \mathcal{D}^2R^2
\end{tikzcd}
$$

$$\mathrm{grad}\, f = \nabla f = \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y}$$

$$\mathrm{rot}\, V = \nabla \times \left(v^x \frac{\partial}{\partial x} + v^y \frac{\partial}{\partial y}\right) = \frac{\partial v^y}{\partial x} - \frac{\partial v^x}{\partial y}$$

$$\mathrm{div}\, V = \nabla \cdot \left(v^x \frac{\partial}{\partial x} + v^y \frac{\partial}{\partial y}\right) = \frac{\partial v^x}{\partial x} + \frac{\partial v^y}{\partial y}$$

$$V = v^x \frac{\partial}{\partial x} + v^y \frac{\partial}{\partial y} \xleftrightarrow{\;1\;} v^x dx + v^y dy \ \ (\text{t.c. } W \mapsto V \cdot W)$$

$$V = v^x \frac{\partial}{\partial x} + v^y \frac{\partial}{\partial y} \xleftrightarrow{\;1'\;} v^x dy - v^y dx \ \ (\text{t.c. } W \mapsto \omega_{R^2}(V, W))$$

$$f \xleftrightarrow{\;2\;} f\,\omega_{R^2} = f\, dx \wedge dy$$

Nota: $\mathrm{rot}(\mathrm{grad}\, f) = 0$

Corol. $C \subset R^2$ arco regolare orientato da p a q
$f \in C^\infty(R^2) \Rightarrow \int_C \langle \mathrm{grad}\, f, T\rangle\, ds = f(q) - f(p)$

Dim. $\int_C \langle \mathrm{grad}\, f, T\rangle\, ds = \int_C f'(s)\, ds = \int_C df = \int_{\partial C} f = f(q) - f(p)$

Corol. $C \subset R^2$ chiuso ammissibile compatto
$V \in \mathcal{V}R^2 \Rightarrow \int_C \mathrm{rot}\, V\, \omega_{R^2} = \int_{\partial C} \langle V, T\rangle\, ds$

Dim. $\int_C \mathrm{rot}\, V\, \omega_{R^2} = \int_C (\partial v^y/\partial x - \partial v^x/\partial y)\, dx \wedge dy$
$= \int_C d(v^x dx + v^y dy) = \int_{\partial C} v^x dx + v^y dy = \int_{\partial C} \langle V, T\rangle\, ds$

Corol. $C \subset R^2$ chiuso ammissibile compatto
$V \in \mathcal{V}R^2 \Rightarrow \int_C \mathrm{div}\, V\, \omega_{R^2} = -\int_{\partial C} \langle V, N\rangle\, ds$

Dim. $\int_C \mathrm{div}\, V\, \omega_{R^2} = \int_C (\partial v^x/\partial x + \partial v^y/\partial y)\, dx \wedge dy$
$= \int_C d(v^x dy - v^y dx) = \int_{\partial C} v^x dy - v^y dx = -\int_{\partial C} \langle V, N\rangle\, ds$

Calcolo vettoriale in R^3

$$\begin{array}{ccccccc}
C^\infty(R^3) & \xrightarrow{\text{grad}} & \mathcal{V}R^3 & \xrightarrow{\text{rot}} & \mathcal{V}R^3 & \xrightarrow{\text{div}} & C^\infty(R^3) \\
\Big\| & & \cong\Big\updownarrow 1 & & \cong\Big\updownarrow 2 & & \cong\Big\updownarrow 3 \\
\mathcal{D}^0R^3 & \xrightarrow[d]{} & \mathcal{D}^1R^3 & \xrightarrow[d]{} & \mathcal{D}^2R^3 & \xrightarrow[d]{} & \mathcal{D}^3R^3
\end{array}$$

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial}{\partial z}$$

$$\begin{aligned}\operatorname{rot} V &= \nabla \times \Big(v^x\frac{\partial}{\partial x} + v^y\frac{\partial}{\partial y} + v^z\frac{\partial}{\partial z}\Big)\\
&= \Big(\frac{\partial v^z}{\partial y} - \frac{\partial v^y}{\partial z}\Big)\frac{\partial}{\partial x} - \Big(\frac{\partial v^z}{\partial x} - \frac{\partial v^x}{\partial z}\Big)\frac{\partial}{\partial y} + \Big(\frac{\partial v^y}{\partial x} - \frac{\partial v^x}{\partial y}\Big)\frac{\partial}{\partial z}\end{aligned}$$

$$\operatorname{div} V = \nabla \cdot \Big(v^x\frac{\partial}{\partial x} + v^y\frac{\partial}{\partial y} + v^z\frac{\partial}{\partial z}\Big) = \frac{\partial v^x}{\partial x} + \frac{\partial v^y}{\partial y} + \frac{\partial v^z}{\partial z}$$

$$V = v^x\frac{\partial}{\partial x} + v^y\frac{\partial}{\partial y} + v^z\frac{\partial}{\partial z} \overset{1}{\longleftrightarrow} v^x dx + v^y dy + v^z dz$$
$$\text{(t.c. } W \mapsto V\cdot W)$$

$$V = v^x\frac{\partial}{\partial x} + v^y\frac{\partial}{\partial y} + v^z\frac{\partial}{\partial z} \overset{2}{\longleftrightarrow} v^x dy\wedge dz - v^y dx\wedge dz + v^z dx\wedge dy$$
$$\text{(t.c. } (W_1, W_2) \mapsto \omega_{R^3}(V, W_1, W_2))$$

$$f \overset{3}{\longleftrightarrow} f\,\omega_{R^3} = f\,dx\wedge dy\wedge dz$$

Note: 1) $\operatorname{rot}(\operatorname{grad} f) = 0$, $\operatorname{div}(\operatorname{rot} V) = 0$

2) $V \overset{1}{\longleftrightarrow} \lambda$, $W \overset{1}{\longleftrightarrow} \mu \Rightarrow V \times W \overset{2}{\longleftrightarrow} \lambda\wedge\mu$

Corol. $C \subset R^3$ arco regolare orientato da p a q

$f \in C^\infty(R^3) \Rightarrow \int_C \langle \operatorname{grad} f, T\rangle\, ds = f(q) - f(p)$

Dim. $\int_C \langle \operatorname{grad} f, T\rangle\, ds = \int_C f'(s)\, ds = \int_C df = \int_{\partial C} f = f(q) - f(p)$

Corol. $C \subset S$ chiuso ammis. comp. in $S \subset R^3$ superf. diff. orientata

$V \in \mathcal{V}R^3 \Rightarrow \int_C \langle \operatorname{rot} V, N\rangle\, dS = \int_{\partial C} \langle V, T\rangle\, ds$

Dim. $\int_C \langle \operatorname{rot} V, N\rangle\, dS = \int_C \langle \operatorname{rot} V, X_1 \times X_2\rangle\, dt^1\wedge dt^2$

$$= \int_C \Big(\frac{\partial v^y}{\partial x} - \frac{\partial v^x}{\partial y}\Big) dx\wedge dy + \Big(\frac{\partial v^z}{\partial x} - \frac{\partial v^x}{\partial z}\Big) dx\wedge dz + \Big(\frac{\partial v^z}{\partial y} - \frac{\partial v^y}{\partial z}\Big) dy\wedge dz$$

$$= \int_C d(v^x dx + v^y dy + v^z dz)$$
$$= \int_{\partial C} v^x dx + v^y dy + v^z dz = \int_{\partial C} \langle V, T \rangle \, ds$$

Corol. $C \subset R^3$ chiuso ammissibile compatto

$$V \in \mathcal{V}R^3 \Rightarrow \int_C \operatorname{div} V \, \omega_{R^3} = \int_{\partial C} \langle V, N \rangle \, dS$$

Dim. $\int_C \operatorname{div} V \, \omega_{R^3} = \int_C (\partial v^x/\partial x + \partial v^y/\partial y + \partial v^z/\partial z) \, dx \wedge dy \wedge dz$

$$= \int_C d(v^x dy \wedge dz - v^y dx \wedge dz + v^z dx \wedge dy)$$
$$= \int_{\partial C} v^x dy \wedge dz - v^y dx \wedge dz + v^z dx \wedge dy$$
$$= \int_{\partial C} \langle V, X_1 \times X_2 \rangle \, dt^1 \wedge dt^2 = \int_{\partial C} \langle V, N \rangle \, dS$$